A numerical method to solve generalized Euler equations.

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Mots-clefs : Optimization, Optimal Transport, Kullback-Leibler, Bregman, Alternate projections, Entropic Method, Fluid Dynamics, Euler equations

In this talk we present a numerical method to solve Brenier’s variational models for incompressible Euler equations [1]. These models give rise to a relaxation in the space of measure-preserving plans of Arnold’s interpretation in terms of geodesics. The relaxation of Euler equations proposed by Brenier can be understood as requiring the resolution of a multi-marginal transportation with an infinite number of marginals. When discretizing Brenier’s problem with $K$ steps in time, one thus faces the resolution of a $K$ marginals OT problem:

$$\inf \int \sum_{i=1}^{K-1} \frac{1}{K} |x_{i+1} - x_i|^2 d\gamma(x_1, \ldots, x_K)$$

s.t. $\gamma \geq 0$, $\int \gamma = 1$, $(e_i)_{\sharp} \gamma = \text{Leb}, i = 1, \ldots, K$, $(e_1, e_K)_{\sharp} \gamma = (s_\ast, s_\ast)_{\sharp} \text{Leb}$,

where the constraints $(e_i)_{\sharp} \gamma = \text{Leb}$ stand for the incompressibility of the fluid ($\text{Leb}$ is the Lebesgue measure on $[0,1]^d$) and $(e_1, e_K)_{\sharp} \gamma = (s_\ast, s_\ast)_{\sharp} \text{Leb}$ expresses that moving fluid particles from the initial configuration $s_\ast$ (one usually has $s_\ast = \text{Id}$) to the final one $s_\ast$ yields equivalent transport plans. We regularise problem (1) by adding an entropy term $E(\gamma) = \int \gamma (\log(\gamma) - 1)$ and then, once discretised also in space, we can re-write (1) as the minimization of the Kullback-Leibler distance. The new problem can be solved by using an alternate projections algorithm as the one detailed in [2]. Finally, we present some numerical results for different final configurations $s_\ast$ in dimension $d \geq 1$. As an example, in figure 1 we present numerical results for the Beltrami flow in $\mathbb{R}^2$: we plot the evolution of particles which at the initial time were in $[0,1/3] \times [0,1]$ (red), $(1/3, 2/3) \times [0,1]$ (green) and $(2/3, 1] \times [0,1]$ (blue).

![Particles at different time steps](image)

Figure 1: Particles at different time steps. The final $T$ is 0.9 and the number of marginals $N$ is 16.

Références
