A numerical method to solve generalized Euler equations.

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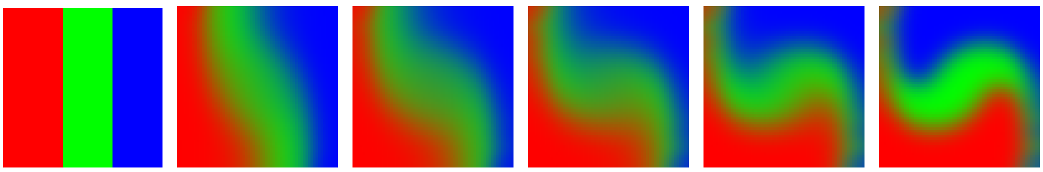
In this talk we present a numerical method to solve Brenier's variational models for incompressible Euler equations [1]. These models give rise to a relaxation in the space of measure-preserving plans of Arnold's interpretation in terms of geodesics. The relaxation of Euler equations proposed by Brenier can be understood as requiring the resolution of a multi-marginal transportation with an infinite number of marginals. When discretizing Brenier's problem with K steps in time, one thus faces the resolution of a K marginals OT problem:

$$\inf \int \sum_{i=1}^{K-1} \frac{1}{K} |x_{i+1} - x_i|^2 d\gamma(x_1, \cdots, x_K) \tag{1}$$

$$s.t. \quad \gamma \geq 0, \quad \int \gamma = 1, \quad (e_i)_\sharp \gamma = Leb, i = 1, \cdots, K \quad (e_1, e_K)_\sharp \gamma = (s_\star, s^\star)_\sharp Leb, \tag{2}$$

where the constraints $(e_i)_\sharp \gamma = Leb$ stand for the incompressibility of the fluid (Leb is the Lebesgue measure on $[0,1]^d$) and $(e_1, e_K)_\sharp \gamma = (s_\star, s^\star)_\sharp Leb$ expresses that moving fluid particles from the initial configuration $s_\star$ (one usually has $s_\star = Id$) to the final one $s^\star$ yields equivalent transport plans. We regularise problem (1) by adding an entropy term $\mathcal{E}(\gamma) = \int \gamma(\log(\gamma) - 1)$ and then, once discretised also in space, we can re-write (1) as the minimization of the Kullback-Leibler distance. The new problem can be solved by using an alternate projections algorithm as the one detailed in [2]. Finally, we present some numerical results for different final configurations $s^\star$ in dimension $d \geq 1$. As an example, in figure 1 we present numerical results for the Beltrami flow in $\mathbb{R}^2$: we plot the evolution of particles which at the initial time were in $[0, 1/3] \times [0, 1]$ (red), $(1/3, 2/3] \times [0, 1]$ (green) and $(2/3, 1] \times [0, 1]$ (blue).



$t = 0$ $\quad t = \frac{4}{N}T$ $\quad t = \frac{7}{N}T$ $\quad t = \frac{10}{N}T$ $\quad t = \frac{13}{N}T$ $\quad t = T$

Figure 1: *Particles at different time steps.* The final T is 0.9 and the number of marginals N is 16.

Références

[1] Y. Brenier, *Journal of American Mathematical Society. The least action principle and the related concept of generalized flows for incompressible perfect fluids*, 1989.

[2] J.D. Benamou, G. Carlier, M. Cuturi, L. Nenna, G. Peyré, *SIAM Journal on Scientific Computing. Iterative bregman projections for regularized transportation problems*, Society for Industrial and Applied Mathematics, 37, 2, A1111-A1138, 2015.