Semidefinite Approximations of Reachability Sets for Discrete-time Polynomial Systems

Victor MAGRON
CNRS VERIMAG, 2 avenue de Vignates 38610 Gières France.

Pierre-Loïc GAROCHE
Onera, the French Aerospace Lab, France.

Didier HENRION
LAAS-CNRS, 7 avenue du colonel Roche, 31400 Toulouse France.

Xavier THIRIOUX
IRIT, avenue de l'étudiant, 31400 Toulouse France.

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Given a compact semialgebraic set $\mathbf{X}_0 \subseteq \mathbb{R}^n$, a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$, we consider the problem of approximating the reachability set of the discrete-time polynomial system $\mathbf{x}_{k+1} = f(\mathbf{x}_k), k = 0, 1, 2, \ldots, \mathbf{x}_0 \in \mathbf{X}_0$, defined as the set $\mathbf{X}^* = \mathbf{X}_0 \cup f(\mathbf{X}_0) \cup f(f(\mathbf{X}_0)) \cup \cdots$. Note that $\mathbf{X}^*$ is typically non-convex and non-connected, even in the case when $\mathbf{X}_0$ is convex and f is linear.
Assuming that $\mathbf{X}^* \subseteq \mathbf{X}$, with $\mathbf{X} \subseteq \mathbb{R}^n$ being a "simple" set (box or ellipsoid), we provide a method to compute certified outer approximations of $\mathbf{X}^*$. This method can be seen as an extension of previous works: [2] where the authors consider the problem of approximating the volume of a compact semialgebraic set, [3] where the authors approximate the image set of a compact semialgebraic set and [1] where the authors approximate the region of attraction of a controlled polynomial system subject to compact semialgebraic constraints.
Here, the proposed method consists of building a hierarchy of relaxations for the infinite dimensional moment problem whose optimal value is the volume of $\mathbf{X}^*$ and whose optimum is the restriction of the Lebesgue measure on $\mathbf{X}^*$. Then, one can outer approximate $\mathbf{X}^*$ as closely as desired with a hierarchy of super level sets of the form $\mathbf{X}^r := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) \geq 0\}$, for some polynomials $v_r \in \mathbb{R}[\mathbf{x}]$ of increasing degrees $2r$.
For each fixed r, finding the coefficients of the polynomial v_r boils down to computing the optimal solution of a semidefinite program. We provide guarantees of strong convergence to $\mathbf{X}^*$ in $L_1(\mathbf{X})$-norm, when the degree of the polynomial approximation tends to infinity and prove in particular that $\lim_{r\to\infty} \mathrm{vol}(\mathbf{X}^r \backslash \mathbf{X}^*) = 0$. We also present some application examples together with numerical results.

Références

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