About the the minimal time crisis problem

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Given a non-empty closed subset of $\mathbb{R}^n$, we consider the *time crisis problem* introduced in [3] which amounts to minimize the functional

$$\tau(x_0) := \int_0^{+\infty} \mathbf{1}_{K^c}(x_u(t))\ \mathrm{d}t, \tag{1}$$

among solutions $x_u(\cdot)$ of the Cauchy problem

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in \mathbb{R}_+ \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

Here $\mathbf{1}_{K^c}(\cdot)$ denotes the characteristic function of the set K^c, hence (1) represents the total time $\tau(x_0) \in [0, +\infty]$ spent by a solution of (2) in the *crisis set* K^c. Based on a regularization method [1] and on a hybrid maximum principle [2], we provide necessary optimality conditions for the time crisis problem. We show that optimal trajectories for the regularized problem converge to an optimal solution of the time crisis problem. We also study convergence of extremal trajectories and of the adjoint vector satisfying Pontryagin's Principle for the regularized problem. When the viability kernel of K is empty and if trajectories can enter and leave K an infinite number of times, then an interesting question is to find an optimal feedback control minimizing (1) over a finite horizon. We study this question for a chemostat model with two species [4] described by the following controlled system:

$$\begin{cases} \dot{x}_1(t) = f_1(x_3(t))x_1(t) - u(t)x_1(t), \\ \dot{x}_2(t) = f_2(x_3(t))x_2(t) - u(t)x_2(t), \\ \dot{x}_3(t) = -f_1(x_3(t))x_1(t) - f_2(x_3(t))x_2(t) + u(t)(1 - x_3(t))). \end{cases} \tag{3}$$

The set K is given by $K := \{(x_1, x_2, s) \in \mathbb{R}^3_+ \ ; x_1 \geq \underline{x}_1\}$ where $\underline{x}_1$ represents a threshold such that the viability kernel of K is empty. This problem finds typical applications when one would like to harvest species 1 in an environment where it is non necessarily dominant with respect to the other species. We show the existence of periodic trajectories that enter and leave K an arbitrary number of times and we compute numerically an optimal control using the regularization scheme.

Références

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