Smoothness of the Metric Projection onto Nonconvex Sets in Hilbert Spaces

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In his fundamental paper from 1973 [1], R. B. Holmes showed that, whenever we have a convex closed set K in a Hilbert space X such that

(i) K has nonempty relative interior (namely, the interior of K as a subset of $Y = \overline{\mathrm{aff}}(K)$ is nonempty), and

(ii) the boundary of K as a subset of Y, $\mathrm{bd}\, K$, is a $\mathcal{C}^{p+1}$-submanifold near a point $x_0 \in \mathrm{bd}\, K$,

then the metric projection P_K is a function of class $\mathcal{C}^p$ in an open neighborhood W of the open normal ray $R_{x_0}(K) = \{x_0 + t\nu \ :\ t > 0\}$, where ν denotes the unit exterior normal vector of K at x_0. His main observations to arrive to this theorem were:

1. It is enough to prove the theorem for convex bodies (namely, where K has nonempty interior), since under (i), we can write
$$P_K = \left(P_K\big|_Y\right) \circ \pi_Y,$$
where π_Y denotes the orthogonal projection to Y (which is a continuous linear function and therefore of class $\mathcal{C}^\infty$);

2. The smoothness of $\mathrm{bd}\, K$ at x_0 can be translated as the smoothness of the Minkowski functional ρ_K (independently of which translation is used to ensure that 0 is an interior point of K); furthermore, the equality $\nu = \|\nabla\rho_K(x_0)\|^{-1}\nabla\rho_K(x_0)$ holds true;

3. The distance function d_K is of class $\mathcal{C}^1$ in $X \setminus K$; and finally,

4. For any point $x \in R_{x_0}(K)$ and a suitable choice of neighborhoods U and V of x and x_0 respectively, the mapping
$$F : U \times V \to X$$
$$(u, v) \mapsto u - v - d_K(u)\frac{\nabla\rho_K(u)}{\|\nabla\rho_K(u)\|}$$
is well defined, of class $\mathcal{C}^1$, and for every $(u, v) \in U \times V$, one has
$$F(u, v) = 0 \iff v = P_K(u).$$

With all these observations, he concluded his theorem through an application of the well-known Implicit Function Theorem. Following the way opened by the strategy of Holmes, the aim of the present work is to establish, under the same hypotheses (i)-(ii), similar *local* results dropping the hypothesis of convexity and replacing it with *prox-regularity.*

The main motivation for this research came from the huge advances made in Proximal Analysis and from the 2000's paper by Poliquin, Rockafellar and Thibault [2], which allows us to replace the continuous differentiability of the distance function to convex bodies, with another suitable one related to prox-regular sets. Also we want to mention Mazade Ph. D. Thesis [4], on which local prox-regularity was profoundly studied in a quantified sense.

Références

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