Learning in Mean Field Games: The Fictitious Play

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Mean Field Game is a class of differential games in which each agent is infinitesimal and interacts with a huge population of other agents. These games have been introduced simultaneously by Lasry, Lions [3, 4] and Huang, Malhamé and Caines [2]. The classical notion of equilibrium solution in Mean Field Game (abbreviated MFG) is given by a pair of maps (u, m), where $u = u(t, x)$ is the value function of a typical small player while $m = m(t, x)$ denotes the density at time t and at position x of the population.
Our aim is to define a Fictitious Play for the MFG system and to prove the convergence of this procedure under suitable assumption on the couplings f and g. This yields to define by induction the sequences $u^n, m^n, \bar{m}^n$ by:

$$\begin{cases} (i) & -\partial_t u^{n+1} - \sigma\Delta u^{n+1} + H(x, \nabla u^{n+1}(t,x)) = f(x, \bar{m}^n(t)), \\ (ii) & \partial_t m^{n+1} - \sigma\Delta m^{n+1} - \operatorname{div}(m^{n+1} D_p H(x, \nabla u^{n+1})) = 0, \\ & m^{n+1}(0) = m_0,\ u^{n+1}(x,T) = g(x, \bar{m}^n(T)) \end{cases} \tag{1}$$

where $\bar{m}^n = \frac{1}{n}\sum_{k=1}^n m^k$. Indeed, u^{n+1} is the value function at stage $n+1$ if the belief of players on the evolving density is $\bar{m}^n$, and thus solves (1)-(i). The actual density then evolves according to the Fokker-Planck equation (1)-(ii).
Our main result is that, under suitable assumption, this learning procedure converges, i.e., any cluster point of the pre-compact sequence (u^n, m^n) is a solution of the MFG system.

References

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