Exact algorithms for linear matrix inequalities

Mohab Safey El Din
(UPMC/CNRS/IUF/INRIA PolSys Team)

Joint work with
Didier Henrion, CNRS LAAS (Toulouse)
Simone Naldi, Technische Universität Dortmund

SMAI-MODE, 2016
Spectrahedra and LMI

$A_0, A_1, \ldots, A_n$ are $m \times m$ real symmetric matrices

Spectrahedron: $\mathcal{S} = \{ x \in \mathbb{R}^n : A(x) = A_0 + x_1A_1 + \cdots + x_nA_n \succeq 0 \}$

It is **basic semi-algebraic** since, if

$$\det(A(x) + tl_m) = f_m(x) + f_{m-1}(x)t + \cdots + f_1(x)t^{m-1} + t^m$$

then $\mathcal{S} = \{ x \in \mathbb{R}^n : f_i(x) \geq 0, \ i = 1, \ldots, m \}$. $A(x) \succeq 0$ is called an **LMI**.
Spectrahedra and LMI

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then \( \mathcal{S} = \{ x \in \mathbb{R}^n : f_i(x) \geq 0, \ i = 1, \ldots, m \} \). \( A(x) \succeq 0 \) is called an LMI.

SDP: linear optimization over \( \mathcal{S} \) (i.e. over LMI)

\[
\mathcal{S} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \succeq 0 \right\}
\]

Figure: The Cayley spectrahedron
Why exact algorithms?

1. It is **Hard** to compute low-rank solutions to SDP

Figure: “Low-rank” points: they minimize a cone of linear forms

Figure: SEDUMI returns a floating point approximation of \((0, 0)\) when maximizing \(x_2\)

2. The interior of \(\mathcal{S}\) can be **empty** \(\rightarrow\) **Interior point algorithms could fail**

\[
\begin{bmatrix}
0 & x_1 & \frac{1}{2}(1 - x_4) \\
x_1 & x_2 & x_3 \\
\frac{1}{2}(1 - x_4) & x_3 & x_4
\end{bmatrix} \succeq 0
\]
Why exact algorithms?

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Main motivations for the design of **exact algorithms**:

1. Can we manage **algebraic constraints** such as rank defects?
2. Can we handle **degenerate** non-full-dimensional examples?
3. **Consequence:**
   The output is a point whose coordinates may be **real algebraic numbers**

$$ (q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[t] \rightarrow \left\{ \left( \frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)} \right) : q(t) = 0 \right\} $$
State of the art

*Decision/Sampling problem for real algebraic or semi-algebraic sets*

**Cylindrical Algebraic Decomposition**

- Tarski (1948), Seidenberg, Cohen, …
- Collins (1975) in $O((2m)^{2n+8} m^{2n+6})$, …

**Critical Points Method**

- Grigoriev, Vorobjov (1988) first singly exp: $m^{O(n^2)}$
  - linear exponent $m^{O(n)}$

**Polar varieties**

- Bank, Giusti, Heintz, Mbakop, Pardo (1997, …)
- Safey El Din, Schost (2003, 2004) regular in $O(m^{3n})$, singular in $O(m^{4n})$

The goal was:

**Better results for spectrahedra?**
**How to take advantage of the structure?**
## Complexity of SDP

### Special case of SDP

Khachiyan, Porkolab (1996) decide LMI-feasibility in time

\[
O(nm^4) + m^{O(\min\{n,m^2\})} \quad \text{on} \quad (\ell m^{O(\min\{n,m^2\})})\text{-bit numbers}
\]

\(\ell = \text{input bit-size}\)

---

**Positive aspects:**
1. No assumptions, Deterministic
2. Binary complexity

**Main drawbacks:**
1. It relies on Quantifier Elimination
2. Too large constant in the exponent
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Low rank positive semidefinite matrices

Define:

For any $A(x)$ (not nec. symmetric): $\mathcal{D}_r = \{x \in \mathbb{C}^n : \text{rank } A(x) \leq r\}$

For $A(x)$ symmetric, and $\mathcal{S} \neq \emptyset$: $r(A) = \min \{\text{rank } A(x) \mid x \in \mathcal{S}\}$

So one has nested sequences

\[
\mathcal{D}_0 \subset \cdots \subset \mathcal{D}_{m-1} \\
\mathcal{D}_0 \cap \mathbb{R}^n \subset \cdots \subset \mathcal{D}_{m-1} \cap \mathbb{R}^n
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Smallest Rank Property  |  Henrion-N.-Safey El Din 2015
--- | ---
\( A(x) \) symmetric, and \( \mathcal{I} \neq \emptyset \). Let \( \mathcal{C} \) be a conn. comp. of \( \mathcal{D}_{r(A)} \cap \mathbb{R}^n \) s.t. \( \mathcal{C} \cap \mathcal{I} \neq \emptyset \). Then \( \mathcal{C} \subset \mathcal{I} \). In particular \( \mathcal{C} \subset \mathcal{D}_{r(A)} \setminus \mathcal{D}_{r(A)-1} \).
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Smallest Rank Property: Henrion-N.-Safey El Din 2015

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Either
the set $\mathcal{I}$ is empty
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So one has nested sequences

$$D_0 \subset \cdots \subset D_{m-1}$$

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Either

the set $\mathcal{I}$ is empty

Or

$r(A)$ is well-defined

$\exists C \subset D_{r(A)} : C \subset \mathcal{I}$
Low rank positive semidefinite matrices

A(x) symmetric, and I ≠ ∅. Let C be a conn. comp. of D_{r(A)} ∩ R^n s.t. C ∩ I ≠ ∅. Then C ⊂ I. In particular C ⊂ D_{r(A)} \ D_{r(A)−1}.

Either
the set I is empty

Or
r(A) is well-defined
∃ C ⊂ D_{r(A)} : C ⊂ I
Problem statement

Emptiness of spectrahedra

Given $A(x)$ symmetric, with entries in $\mathbb{Q}$, compute a finite set meeting $\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$, or establish that $\mathcal{S}$ is empty.

In other words: Decide the feasibility of an LMI $A(x) \succeq 0$.

Particular instance of: Decide the emptiness of semi-algebraic sets.
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Sample points on $\mathcal{S}$ + Smallest Rank Property = Sample points on $\mathcal{D}_{r(A)} \cap \mathbb{R}^n$
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Sample points on $\mathcal{S}$ + Smallest Rank Property = Sample points on $\mathcal{D}_r(A) \cap \mathbb{R}^n$

Real root finding on determinantal varieties

Given any $A(x)$ with entries in $\mathbb{Q}$, compute a finite set meeting each connected component of $\mathcal{D}_r \cap \mathbb{R}^n = \{x \in \mathbb{R}^n : \text{rank} A(x) \leq r\}$.

Particular instance of: Sampling real algebraic sets.
1. The **Smallest Rank Property** \( \exists C \subset D_r(A) : C \subset \mathcal{S} \) allows to reduce:

| Sampling/Optimization over One semi–algebraic set | \( \rightarrow \) | Sampling/Optimization over Many algebraic sets |

This is somehow *typical* in PO. Ex. Polar Varieties for PO: **Safety El Din, Greuet**
Strategy

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   But in our case the **structure is preserved**!
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2. For \(r = 1, \ldots, m - 1\) compute sample points in \(D_r \cap \mathbb{R}^n\)

3. Output the minimum rank on \(\mathcal{I}\) with a sample point.
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**Sampling determinantal varieties**

- Either the empty list iff
  \[ \mathcal{D}_r \cap \mathbb{R}^n = \emptyset \]
- Or \((q, q_1, \ldots, q_n) \subset \mathbb{Q}[t]\) s.t.
  \[ \forall C \subset \mathcal{D}_r \cap \mathbb{R}^n \exists t : x(t) \in C \]
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Incidence varieties and critical points

1st step  *Lifting of the determinantal variety*:

\[ A(x) Y(y) = A(x) \begin{bmatrix} y_{1,1} & \cdots & y_{1,m-r} \\ \vdots & \ddots & \vdots \\ y_{m,1} & \cdots & y_{m,m-r} \end{bmatrix} = 0. \]

\[ U Y(y) = I_{m-r} \]

If \( A \) is generic, the lifted algebraic set \( \mathcal{V}_r \) is **smooth** and **equidimensional**

2nd step  *Compute critical points* of the map \( \pi(x, y) = a_1 x_1 + \cdots + a_n x_n \) on \( \mathcal{V}_r \):

When \( a_1 .. a_n \) are generic, there are **finitely many** critical points.

3rd step  *Intersect with any fiber of \( \pi \) and call point 2*
Incidence varieties and critical points

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\vdots & & \vdots \\
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\end{bmatrix} = 0.
\]

\[
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\]

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3rd step **Intersect with any fiber of \( \pi \) and call point 2**
Computing critical points on incidence varieties

\[ A(X)Y = 0, \quad UY = I_{m-r} \]

\[ F_1(X, Y) = \cdots = F_{m(m-r)}(Y) = \cdots = F_{m(m-r)+(m-r)^2} = 0 \]

Lagrange System \( L\text{.jac}(F, [X, Y]) = [a \ 0] \)
Computing critical points on incidence varieties

Multi-Linear System

- \( F(X, Y) = 0, \ G(X, L) = 0, \ H(Y, L) = 0 \)
- All equations have multi-degree \((1, 1, 0)\) or \((1, 0, 1)\) or \((0, 1, 1)\)
- Impact on multi-linear/sparsity structure on the number of solutions

Multi-linear Bézout bounds

- Symbolic-homotopy algorithms
  \( \leadsto \) cubic complexity in these bounds

Symbolic Newton iteration
  (Hensel lifting)

Jeronimo/Matera/Solerno/Waissbein
Complexity bounds

Complexity for Sampling determinantal varieties

\[ O^\sim \left( (n + m^2 - r^2)^7 \left( n + m(m - r) \right)^6 \right) \]

Complexity for Emptiness of spectrahedra

\[ O^\sim \left( n \sum_{r \leq r(A)} \binom{m}{r} (n + p_r + r(m - r))^7 \left( \frac{p_r + n}{n} \right)^6 \right) \]

\[ O^\sim(k) = O(k \log^c k) \quad \exists c \in \mathbb{N} \quad \text{with } p_r = (m - r)(m + r + 1)/2. \]

Remarkable aspects:

- Explicit constants in the exponent
- When \( m \) is fixed, polynomial in \( n \)
- Strictly depends on \( r(A) \)
SPECTRA: a library for real algebraic geometry and optimization

What is SPECTRA?

A MAPLE library, freely distributed
Depends on Faugère’s FGB for computations with Gröbner bases
Addressed to researchers in Optimization, Convex alg. geom., Symb. comp.

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<th>SPECTRA</th>
<th>deg</th>
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- RAGLIB = Real algebraic geometry library
- SPECTRA = new algorithms
- deg = degree of Rational Parametrization
- Time in seconds
- ∞ = more than 2 days

Download a beta version: homepages.laas.fr/snaldi/software.html
Scheiderer’s spectrahedron

\[ f = u_1^4 + u_1 u_2^3 + u_2^4 - 3 u_1^2 u_2 u_3 - 4 u_1 u_2^2 u_3 + 2 u_1^2 u_3^2 + u_1 u_3^3 + u_2 u_3^3 + u_3^4 \]

One can write \( f = v' A(x) v \) with \( v = [u_1^2, u_1 u_2, u_2^2, u_1 u_3, u_2 u_3, u_3^2] \)

\[
A(x) = \begin{bmatrix}
1 & 0 & x_1 & 0 & -3/2 - x_2 & x_3 \\
0 & -2x_1 & 1/2 & x_2 & -2 - x_4 & -x_5 \\
x_1 & 1/2 & 1 & x_4 & 0 & x_6 \\
0 & x_2 & x_4 & -2x_3 + 2 & x_5 & 1/2 \\
-3/2 - x_2 & -2 - x_4 & 0 & x_5 & -2x_6 & 1/2 \\
x_3 & -x_5 & x_6 & 1/2 & 1/2 & 1
\end{bmatrix}
\]

What information can be extracted?

- No matrices of rank 1 s.t. \( A(x) \succeq 0 \rightarrow f \neq g^2 \)
- Two matrices of rank 2 s.t. \( A(x) \succeq 0 \rightarrow f = g_1^2 + g_2^2 = g_3^2 + g_4^2 \)
- No matrices of rank 3 s.t. \( A(x) \succeq 0 \rightarrow f \neq h_1^2 + h_2^2 + h_3^2 \)
Perspectives

1. Remove genericity assumptions on the input linear matrix $A$

2. Use of numerical homotopy for studying incidence varieties

3. Theoretical toolbox for analyzing singularities of determinantal varieties
   Surprising applications in optimal control techniques for the contrast imaging problem in medical imagery
   joint work with B. Bonnard, J.-C. Faugère, A. Jacquemard, T. Verron.