Generalized Additive Independence models and $k$-ary capacities in multicriteria decision making

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Multiattribute utility theory (MAUT) is a widely used framework for decision under multiple criteria.
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Introduction

- Multiattribute utility theory (MAUT) is a widely used framework for decision under multiple criteria.
- The most popular models in MAUT are the additive utility model, and the multiplicative model, satisfying (mutual) preferential independence.
- So far, few models take into account interaction between criteria: the Choquet integral model (Lovász extension), and the multilinear model (Owen extension).
- The GAI (Generalized Additive Independence) model generalizes the additive model, does not satisfy preferential independence, and includes as particular cases CI, MLE.
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Aim of the talk: relate the GAI model with $k$-ary capacities.
Framework

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- **Assumption 2**: Boundaries:
  \[
  U(x_i^\top, \ldots, x_n^\top) = 1, \quad U(x_i^\perp, \ldots, x_n^\perp) = 0
  \]
  with $x_i^\top, x_i^\perp$ the best and worst elements of $X_i$ according to $\succ_i$
The GAI (Generalized Additive Independence) model

- **Additive Utility model**

\[ U(x) = u_1(x_1) + \cdots + u_n(x_n) \]
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- A GAI model is **\( p \)-additive** if any set \( S \in \mathcal{S} \) satisfies \( |S| \leq p \). Hence, a 1-additive GAI model is a classical additive utility model.
Capacities and $k$-ary capacities

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- One may then consider $k$-ary capacities (G. and Labreuche 2003) $\nu : \{0, 1, \ldots, k\}^N \rightarrow \mathbb{R}$ (a.k.a. multichoice games, Hsiao and Raghavan 1990):
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Discrete GAI models are $k$-ary capacities

- We consider that attributes are discrete:

\[ X_i = \{ a_{i,0}^i, \ldots, a_{i,m_i}^i \} \]

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- Given a GAI model \( U \) with discrete attributes, we define
  \( \nu : \{0, \ldots, k\}^N \to \mathbb{R} \) by
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  U(x) =: \nu(\varphi(x)) \quad (x \in X)
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- Given a GAI model \(U\) with discrete attributes, we define \(v:\{0, \ldots, k\}^N \to \mathbb{R}\) by
  \[ U(x) =: v(\varphi(x)) \quad (x \in X) \]
  and let \(v(z) := v(m_1, \ldots, m_n)\) when \(z \in \{0, \ldots, k\}^N \setminus \varphi(X)\).

- By assumptions 1 and 2 on \(U\), it follows that \(v\) is a normalized $k$-ary capacity on \(N\).
Let \( \nu : 2^N \rightarrow \mathbb{R} \) be a capacity. Its **Möbius transform** \( m^\nu \) is the (unique) solution of

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\nu(S) = \sum_{T \subseteq S} m^\nu(T)
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- A capacity $\nu$ is (at most) $p$-additive if $m^{\nu}(S) = 0$ whenever $|S| > p$. 

M. Grabisch and Ch. Labreuche ©2016 The GAI model and $k$-ary capacities
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m^\nu(z) = \sum_{y \leq z : z_i - y_i \leq 1 \forall i \in N} (-1)^{\sum_{i \in N}(z_i - y_i)} \nu(y)
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Let \( v : 2^N \to \mathbb{R} \) be a capacity. Its Möbius transform \( m^v \) is the (unique) solution of
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A \( k \)-ary capacity is \textit{(at most) \( p \)-additive} if \( m^v(z) = 0 \) whenever \( |\text{supp}(z)| > p \), where
\[
\text{supp}(z) = \{i \in N \mid z_i > 0\}
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\textbf{Lemma}

Let \(k \in \mathbb{N}\) and \(p \in \{1, \ldots, n\}\). A \(k\)-ary game \(v\) is \(p\)-additive if and only if it has the form

\[
v(z) = \sum_{x \in \{0, \ldots, k\}^N, |\text{supp}(x)| \leq p} v_x(x \land z)
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where \(v_x : \{0, \ldots, k\}^N \to \mathbb{R}\) with \(v_x(0) = 0\).
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\textbf{Lemma}

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It follows that $p$-additive discrete GAI models are $p$-additive $k$-ary capacities (for some $k \in \mathbb{N}$).
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*Given a GAI model, is it always possible to get a decomposition into nonnegative nondecreasing terms?*

We answer this question for 2-additive discrete GAI models (and the answer is: Yes!)
Determining a 2-additive GAI model with $k + 1$ elements in each attribute by learning yields an optimization problem with

$$(k + 1) \binom{n}{1} + (k + 1)^2 \binom{n}{2}$$

unknowns.
Why it is important to solve this problem

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- Moreover, $U$ being nondecreasing, we have
  \[n \times k \times (k + 1)^{n-1}\]
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- Moreover, \( U \) being nondecreasing, we have

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monotonicity conditions to satisfy.

- If a decomposition into nonnegative nondecreasing terms is possible, one has only to check monotonicity of each term. Then the number of monotonicity conditions drops to

\[
n \times k \times [(n - 1)(k + 1) + 1]
\]
Why it is important to solve this problem

Comparison table with $k = 4$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of constraints</td>
<td>2000</td>
<td>75 000</td>
<td>2 500 000</td>
<td>78 125 000</td>
</tr>
<tr>
<td># of constraints with monotone decomposition</td>
<td>256</td>
<td>624</td>
<td>1152</td>
<td>1840</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>12</th>
<th>14</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td># of constraints</td>
<td>2 343 750 000</td>
<td>68 359 375 000</td>
<td>$1.526E+15$</td>
</tr>
<tr>
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<td>2688</td>
<td>3696</td>
<td>7680</td>
</tr>
</tbody>
</table>
The main result

Theorem

Let us consider a 2-additive discrete GAI model $U$ satisfying assumptions 1 and 2. Then there exist nonnegative and nondecreasing functions $u_i : X_i \rightarrow [0, 1]$, $i \in N$, $u_{ij} : X_i \times X_j \rightarrow [0, 1]$, $\{i, j\} \subseteq N$, such that

$$U(x) = \sum_{i \in N} u_i(x_i) + \sum_{\{i, j\} \subseteq N} u_{ij}(x_i, x_j) \quad (x \in X)$$
The problem is equivalent to the decomposition of a 2-additive normalized $k$-ary capacity $\nu$ into a sum of 2-additive $k$-ary capacities whose support has size at most 2.
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**support of \( \nu \):**

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\text{supp}(\nu) = \bigcup_{x \in L^N : m^\nu(x) \neq 0} \text{supp}(x)
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Let $P_{k,2}$ be the polytope of all normalized 2-additive $k$-ary capacities
Sketch of the proof

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- We prove that any vertex of $P_{k,2}$ has support of size at most 2.
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- We prove that any vertex of $\mathcal{P}_{k,2}$ has support of size at most 2.

- Since any $\nu \in \mathcal{P}_{k,2}$ is a convex combination of vertices of $\mathcal{P}_{k,2}$, which are normalized 2-additive $k$-ary capacities, the desired result follows.
Theorem

Let $k \in \mathbb{N}$. The set of extreme points of $\mathcal{P}_{k,2}$, the polytope of normalized 2-additive $k$-ary capacities, is the set of 0-1-valued 2-additive $k$-ary capacities.
Vertices of $\mathcal{P}_{k,2}$

**Theorem**

Let $k \in \mathbb{N}$. The set of extreme points of $\mathcal{P}_{k,2}$, the polytope of normalized 2-additive $k$-ary capacities, is the set of 0-1-valued 2-additive $k$-ary capacities.

**Theorem**

For every $k \in \mathbb{N}$, the size of the support of any 0-1-valued 2-additive $k$-ary capacity is at most 2.
Vertices of $\mathcal{P}_{k,2}$: sketch of the proof

We recall that a matrix $A$ is totally unimodular iff the polyhedron $Ax \leq b$ is integer for every $b$. 
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1. Step 1: the set of vertices of \( P_{k,1} \), (normalized \( k \)-ary capacities) is the set of 0-1-valued \( k \)-ary capacities. Therefore, it remains to prove that any vertex of \( P_{k,2} \) is 0-1-valued.
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▶ Step 1: the set of vertices of $\mathcal{P}_{k,\cdot}$ (normalized $k$-ary capacities) is the set of 0-1-valued $k$-ary capacities. Therefore, it remains to prove that any vertex of $\mathcal{P}_{k,2}$ is 0-1-valued.

▶ Step 2: We prove that $A_{k,\cdot}$, the matrix defining $\mathcal{P}_{k,\cdot}$, is totally unimodular.
Vertices of $\mathcal{P}_{k,2}$: sketch of the proof

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- Step 2: We prove that $A_{k,\cdot}$, the matrix defining $\mathcal{P}_{k,\cdot}$, is totally unimodular.
  
- It follows that the polytope $A_{k,\cdot}x \leq b$ is integer $\forall b$, and so is the polytope $A^m_{k,\cdot}m^\vee \leq b$ for all $b$ (same in the Möbius transform coordinates). Therefore, $A^m_{k,\cdot}$ is also totally unimodular.
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As $A_{k,2}^m$ is a submatrix of $A_{k,\cdot}^m$, it is also totally unimodular. Therefore, the vertices of $\mathcal{P}_{k,2}^m$ are integer-valued.
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We recall that a matrix $A$ is totally unimodular iff the polyhedron $Ax \leq b$ is integer for every $b$.

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- As $A^m_{k,2}$ is a submatrix of $A^m_{k,\cdot}$, it is also totally unimodular. Therefore, the vertices of $\mathcal{P}^m_{k,2}$ are integer-valued.

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- It follows that the polytope $A_{k,2}x \leq b$ is integer $\forall b$, and so is the polytope $A_{k,2}^m m^v \leq b$ for all $b$ (same in the Möbius transform coordinates). Therefore, $A_{k,2}^m$ is also totally unimodular.

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- We prove that $v$ is 0-1-valued iff $m^v$ is $\{-1, 0, 1\}$-valued. The desired result then follows.
Determination of all vertices of $\mathcal{P}_{k,2}$

**Preliminary step:** one shows that the vertices of $\mathcal{P}_{k,2}$ with support included in, say, $\{1, 2\}$, are in bijection with the antichains (which are of size at most $k + 1$) of the lattice $(k + 1)^2$. 
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**Theorem**

Let $k \in \mathbb{N}$ and consider the polytope $\mathcal{P}_{k,2}$. The following holds.

1. For any $i \in \mathbb{N}$, the number of vertices with support $\{i\}$ is $k$.
2. For any distinct $i, j \in \mathbb{N}$, the number of vertices with support included in $\{i, j\}$ is $\binom{2k + 2}{k + 1} - 2$.
3. The total number of vertices of $\mathcal{P}_{k,2}$ is

$$\left[\binom{2k + 2}{k + 1} - 2\right] \frac{n(n - 1)}{2} - kn(n - 2).$$
Any vertex is 0-1-valued and has support of size at most 2, say \{1, 2\}
More details on vertices

- Any vertex is 0-1-valued and has support of size at most 2, say \(\{1, 2\}\)
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- Suppose that \( \text{supp}(v) \subseteq \{1, 2\} \). Denote by \( x^1, \ldots, x^q \) the minimal winning coalitions of \( v \), arranged such that \( x^1 < x^2 < \cdots < x^q \). Then \( m^v(x^\ell) = 1 \) for all \( \ell = 1, \ldots, q \), \( m^v(x^\ell \lor x^{\ell+1}) = -1 \) for \( \ell = 1, \ldots, q - 1 \), and \( m^v(x) = 0 \) otherwise.
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