A new direction in polynomial time interior-point methods for monotone linear complementarity problem

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Outline

1. Problem formulation
2. Interior-point methods
3. New directions
4. Perspectives
The Linear Complementarity Problem (LCP) consists in finding vectors $z \geq 0$ and $s \geq 0$ such that

$$Mz + q = s$$

$$zs = 0,$$

where $zs = (z_is_i)_{1 \leq i \leq n}$.

$(z, s)$ are feasible if they verify $z, s \geq 0$ and $Mz + q = s$.

$M$ satisfies a monotonicity property: all vectors $z \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ that satisfy $Mz - s = 0$ have $z^T s \geq 0$. 

Notation

- $M \in \mathbb{R}^{n \times n}$
- $q \in \mathbb{R}^n$
Linear Complementarity Problem: Applications

Several applications of LCP:
- convex hulls in a plane
- Nash equilibrium in bimatrix games
- absolute value equation
- optimality conditions of optimization problems (Linear, Convex Quadratic)

Monotonicity property of $M$:
- P-matrix (unique solution)
- Positive Semi-Definite (feasible $\Rightarrow$ solvable)
- Skew-Symmetric (Linear Programming)
Interior-Point Method (IPM)

\[ Mz + q = s \]
\[ zs = 0 \]
\[ z, s \geq 0 \]
Given $\mu \geq 0$

\[(LCP)_\mu\]

$Mz + q = s$

$zs = \mu$

$z, s \geq 0$
Given $\mu \geq 0$

$$Mz + q = s$$

$$(LCP)_\mu$$

$$zs = \mu$$

$$z, s \geq 0$$

This system has a unique solution $(z(\mu), s(\mu))$ if the Interior Point Condition holds.

**Interior Point Condition :**

$$\exists z^0, s^0 \text{ such that } s^0 = Mz^0 + q, \; z^0 > 0, \; s^0 > 0$$

$(x(\mu), s(\mu))_\mu$ defines the central path, leading to the optimal solution $(\mu \to 0)$.

IPMs follow the central path approximately.
The most simple IPM: algorithm

Data:
an update parameter $\theta$, $0 < \theta < 1$;

Begin:

$z = z^0$, $s = s^0$, $\mu := \mu^0$;

while $n\mu \geq \epsilon$ do

$\mu := (1 - \theta)\mu$;

$(z, s) := (z, s) + (\Delta z, \Delta s)$;

end

**Algorithm 1**: Full Newton step IPM

$(\Delta z, \Delta s)$ is the unique solution of the system

\[
\begin{align*}
M\Delta z &= \Delta s \\
zs + s\Delta z &= \mu - zs
\end{align*}
\] (1)
The most simple IPM :: illustrations

Complexity for monotone LCP to get $z^T s \leq n\epsilon : \mathcal{O}(\sqrt{n} \log(\frac{n}{\epsilon}))$
where $\theta = \mathcal{O}(\frac{1}{\sqrt{n}})$ has a fixed value.
Given $\mu \geq 0$,\n
\[ Mz + q = s \]
\[ \varphi(zs) = \mu \]
\[ z, s \geq 0 \]

- Introduced in 03’ by Darvay.
- IPM with full Newton step has a complexity in $O(\sqrt{n} \log(\frac{n}{\epsilon}))$ for $\varphi(.) = \sqrt{}$.

**Warning:**

It is not the same as $zs = \varphi^{-1}(\mu)$ since we take a Newton step.
Figure: Level surface of $z_s$
Figure: Level surfaces of $zs$ and $\sqrt{zs}$
Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\varphi(0) = 0$, $\varphi \in C^2$, concave and invertible.

**Figure**: Level surfaces of $zs$ and $\varphi(zs) = \frac{zs}{zs+0.5}$
Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\varphi(0) = 0$, $\varphi \in C^2$, concave and invertible.

**Theorem:**

Let $\bar{\mu} \geq \mu^0 = \frac{(z^0)^T s^0}{n}$. After at most $O(\sqrt{n} \log(\frac{n}{\epsilon}))$ iterations, we have $\varphi(zs)^T e \leq n\epsilon$. The algorithm generates a sequence of update parameter $\theta^k$, guarantees feasibility of the iterates and quadratic convergence of the Newton process.
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Concavity of $\varphi$ gives that for $zs$ sufficiently small

$$\varphi(zs) \approx zs \varphi'(0)$$  \hspace{1cm} (2)
Quadratic convergence of the Newton process

\[ \delta(z^+ s^+, \mu) \leq \delta(z s, \mu)^2 \]

where \( \delta(z s, \mu) \) is a proximity measure.

- classical proximity measure:
  \[
  \frac{1}{2} \left\| \frac{zs}{\mu} - \frac{\mu}{zs} \right\|_2
  \]

- new proximity measure:
  \[
  \frac{1}{2} \left\| \frac{\varphi'(0)}{\varphi'(zs)} \left( \frac{zs}{\mu} - \frac{\mu}{zs} \right) \right\|_2
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  \]
Figure: Level curves for both proximity measures at $\mu^0/\bar{\mu}$ and $\mu^0$
We will consider $\varphi(t) = \frac{t}{t+1}$ so $\varphi'(t) = \frac{1}{(1+t)^2}$.

Observations:
- sequence of update parameters
- central path
- domain of quadratic convergence
Behaviour on an example for LO

The central path and the iterates of both methods. One should note that this figure is presented in the projection of the space of $zs$ in $\mathbb{R}^2$. 
Behaviour on an example for LO

The sequence of update parameter, which converge to its upper bound $\frac{1}{\sqrt{2n+1}}$.
Behaviour on an example for LO

The level curves which guarantees quadratic convergence of the Newton process.
Figure: Step: classical-direction. '*' before Newton, 'o' after Newton.
Behaviour on an example for LO

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Figure: Step: $\varphi$-direction. '*' before Newton, 'o' after Newton.
Behaviour on an example for LO

Figure: Step: \( \varphi \)-direction. '*' before Newton, 'o' after Newton.
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**Figure**: Step: $\varphi$-direction. ‘*’ before Newton, ‘o’ after Newton.
Behaviour on an example for LO

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Condition number in function of $\mu$ (so for our direction it is $\mu/\mu^0$).
Behaviour on an example for LO

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Condition number in function of $\mu$ (so for our direction it is $\mu/\mu^0$).

Behaviour of both system is pretty much the same.
An new IPM method with full Newton step:
- different steps
- polynomial time with the best known bound
- works for a large family of functions $\varphi$
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  \( \rightarrow \) Numerical tests to determine which of those perform best?
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$\rightarrow$ Numerical tests to determine which of those perform best?

And now what?
Embedding technique
Larger system with artificial initial point

Infeasible IPM

\[ s - Mz - q = \nu(s^0 - Mz^0 - q), \quad z, s \geq 0, \quad zs = \mu, \]  

asymptotically feasible. 
Recent developments:

- improved bound for LO [Roos 15']
- \( \varphi(.) = \sqrt{\cdot} \), [Mansouri et al., 14' and 15']
What now? Large Update method

**Large Update method**
- Small Update: takes all the Newton step $\Rightarrow$ find $\theta$.
- Large Update: choose $\theta$ $\Rightarrow$ takes a damped Newton step.

Challenge: ”the irony of IPMs”
- Small-update methods: $O(\sqrt{n} \log(\frac{n}{\epsilon}))$ - inefficient in practice
- Large-update methods: $O(n \log(\frac{\sqrt{n} \log(n)}{\epsilon}))$ - very efficient in practice
What now? A new hobby, relaxation methods for MPCC

Mathematical Program with Complementarity Constraint (MPCC)

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad h_i(x) = 0, \quad i = 1, \ldots, m \\
& \quad g_i(x) \leq 0, \quad i = 1, \ldots, p \\
& \quad 0 \leq G_i(x) \perp H_i(x) \geq 0, \quad i = 1, \ldots, q
\end{align*}
\]
Merci de votre attention !

Figure : Central path on the Klee-Minty cube