APPLICATIONS DU TRANSPORT OPTIMAL DANS LES JEUX À POTENTIEL

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En collaboration avec G. Carlier & P. Mossay & F. Santambrogio

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   - The best reply scheme
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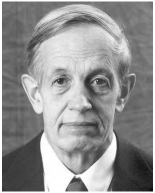
Game theory describes the strategic interactions of rational agents.

PRISONNER'S DILEMMA

- Player 1 chooses the line
- Player 2 chooses the column
- They play simultaneously.
- The pay-off of Player 1 is the first element of the couple, Player 2 is the second.

$$\begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix}$$



Consider N players $i \in \{1, \cdots, N\}$, each player i has a strategy space Y_i. Set

$$Y_{-i} := \Pi_{j \neq i} Y_j \quad \text{and} \quad \forall y \in Y,\ y = (y_i, y_{-i}) \in Y_i \times Y_{-i}\ .$$

The cost function of player i is a function

$$J_i :\ Y_i \times Y_{-i} \to \mathbb{R}.$$

DEFINITION (NASH EQUILIBRIUM)

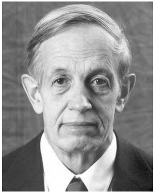
A *Nash-equilibrium* is a collection of strategies $\overline{y} = (\overline{y}_1, \cdots, \overline{y}_N) = (\overline{y}_i, \overline{y}_{-i}) \in Y_i \times Y_{-i}$ such that, for every $i \in \{1, \cdots, N\}$ and every $y_i \in Y_i$, one has:

$$J_i(\overline{y}_i, \overline{y}_{-i}) \leq J_i(y_i, \overline{y}_{-i}).$$

MATCHING PENNY

- Player 1 chooses the line
- Player 2 chooses the column
- They play simultaneously.
- The pay-off of Player 1 is the first element of the couple, Player 2 is the second.

$$\begin{pmatrix} (-1, 1) & (1, -1) \\ (1, -1) & (-1, 1) \end{pmatrix}$$



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$$J_i(\overline{y}_i, \overline{y}_{-i}) \leq J_i(y_i, \overline{y}_{-i}).$$

A mixed strategy for player i is by definition a probability measure $\pi_i \in \mathcal{P}(Y_i)$ and given a profile of mixed strategies $(\pi_1, \cdots \pi_N) \in \Pi_{i=1}^N \mathcal{P}(Y_i)$, the cost for player i reads

$$\overline{J}_i(\pi_1, \cdots \pi_N) := \int_Y J_i(y_1, \cdots y_N) \otimes_{j=1}^N \pi_j(\mathrm{d}y_j).$$

THEOREM (EXISTENCE OF NASH EQUILIBRIA, NASH (1950))

Any game with compact metric strategy spaces and continuous costs has at least one Nash equilibrium in mixed strategies. In particular, finite games admit Nash equilibria in mixed strategies.

VON NEUMANN-MORGENSTERN (1944)

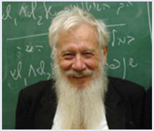
"*It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case.*"

"*When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory become possible.*"

AUMANN (1964)

"*The most natural model for this purpose contains a continuum of participants, similar to the continuum of points on a line or the continuum of particles in a fluid.*"

"The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of a branch of mathematics called "analysis", in a situation where treatment by finite methods would be much more difficult or hopeless."



- Each agent has to choose a strategy y from some strategy compact space Y.
- The cost of one agent depends on the other agents' choice through the probability distribution $\nu \in \mathcal{P}(Y)$ resulting from the whole population strategy choice:

$$\begin{aligned} F : Y \times \mathcal{P}(Y) &\rightarrow \mathbb{R} \\ (y, \nu) &\mapsto F(y, \nu) \end{aligned}$$

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EXAMPLE: BECKMANN'S URBAN REGION MODEL

- Rivalry/Congestion: The cost of the agent increases when the number of players who choose the same action increases. (Consumption of the same public good (motorway game) ; Food supply in an habitat decreases with the number of its users (ex. Sticklebacks (Milinsky)).
- Interactions: The cost of the agents decreases because some other agents play a similar action.

Consider

$$F(y, \nu) := \underbrace{f[\nu(y)]}_{\text{congestion}} + \underbrace{\int_Y \phi(|y - z|)\, \mathrm{d}\nu(z)}_{\text{interaction}} + \underbrace{A(y)}_{\text{amenities}} .$$

where

- f is the competition for land.
- ϕ is the travelling cost.

Cournot-Nash equilibrium: the regular case

The probability $\nu \in \mathcal{P}(Y)$ is a Cournot-Nash equilibrium if:

$$\forall z \in Y,\ F(y,\nu) \leq F(z,\nu)\ .$$

Defining $V := inf_z F(z,\nu)$, at equilibrium we should have

$$\begin{cases} F(y,\nu) = V & \nu\text{-a.e. } y, \\ F(y,\nu) \geq V & \text{a.e. } y \in Y. \end{cases}$$

Cournot-Nash equilibrium

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- Agents are characterised by a type $x \in X$: $\mu \in \mathcal{P}(X)$ gives the distribution of types in the agents population.
- Each agent has to choose a strategy y from some strategy space Y.
- The cost of one agent depends on the other agents' choice through the probability distribution $\nu \in \mathcal{P}(Y)$ resulting from the whole population strategy choice:

$$\begin{aligned} F : X \times Y \times \mathcal{P}(Y) &\to \mathbb{R} \\ (x, y, \nu) &\mapsto F(x, y, \nu) \end{aligned}$$

COURNOT-NASH EQUILIBRIUM

A probability $\gamma \in \mathcal{P}(X \times Y)$ is a *Cournot-Nash equilibrium* if its first marginal is μ, its second marginal ν and there exists $\varphi \in \mathcal{C}(X)$ such that

$$\begin{cases} F(x, y, \nu) \geq \varphi(x) & \forall x \in X \text{ and } m_0\text{-a.e. } y \in Y \\ F(x, y, \nu) = \varphi(x) & \text{for } \gamma\text{-a.e. } (x, y) \in X \times Y \,. \end{cases} \tag{1}$$

A Cournot-Nash equilibrium γ is called *pure* if it is of the form $\gamma = (\mathrm{id}, T)_{\#}\mu$.

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We assume that player i's cost is given by

$$J_i^N(y_i, y_{-i}) = F^N\left(x_i, y_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{y_j}\right)$$

with F^N a sequence of uniformly equi-continuous functions on $X \times Y \times \mathcal{P}(Y)$

THEOREM (NASH EQUILIBRIA CONVERGE TO COURNOT-NASH EQUILIBRIA, B.-CARLIER (2015))

Let $\overline{y}^N = (y_1^N, \cdots, y_N^N)$ be a Nash equilibrium for the game above, and define:

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \quad \nu^N := \frac{1}{N}\sum_{i=1}^N \delta_{y_i^N} \quad \textit{and} \quad \gamma^N := \frac{1}{N}\sum_{i=1}^N \delta_{(x_i, y_i^N)}$$

Assume that, up to the extraction of sub-sequences, $\mu^N \overset{*}{\rightharpoonup} \mu$, $\nu^N \overset{*}{\rightharpoonup} \nu$, $\gamma^N \overset{*}{\rightharpoonup} \gamma$ and $F^N \to F$ in $\mathcal{C}(X \times Y \times \mathcal{P}(Y))$ then γ is a Cournot-Nash equilibrium.
The result also applies to the extension in mixed strategies with the extended cost

$$\overline{J}^N(x_i, \pi_i, \pi_{-i}) := \int_{Y^N} J^N(x_i, y, y_{-i})\, \pi_i(\mathrm{d}y) \otimes_{j\neq i} \pi_j(\mathrm{d}y_j)\ .$$

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THEOREM (EXISTENCE OF COURNOT-NASH EQUILIBRIA, MASCOLELL (1984))

If $F \in \mathcal{C}(X \times Y \times \mathcal{P}(Y))$, there exists at least one Cournot-Nash equilibrium.

RESTRICTION

$f \equiv 0$ (no congestion).

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THEOREM (UNIQUENESS UNDER MONOTONICITY, P.-L. LIONS & J.-M. LASRY (2006))

If $\nu \mapsto \mathcal{V}[\nu]$ is strictly monotone in the sense that for every ν_1 and ν_2 in $\mathcal{D}$, one has

$$\int_X (\mathcal{V}[\nu_1] - \mathcal{V}[\nu_2])\,\mathrm{d}(\nu_1 - \nu_2) \geq 0$$

and the inequality is strict whenever $\nu_1 \neq \nu_2$, then, all the equilibria have the same second marginal.

RESTRICTION

$\phi \equiv 0$ (no interaction).

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SEPARABLE CASE

From now on we assume

$$F(x, y, \nu) = c(x, y) + \mathcal{V}[\nu](y).$$

where

$$c(x, y) = \frac{|x - y|^2}{2} \quad \text{and} \quad \mathcal{V}[\nu](y) = \underbrace{f(\nu(y))}_{\text{congestion}} + \underbrace{\int_Y \phi(y, z)\, \mathrm{d}\nu(z)}_{\text{interaction}}$$

with $f(y) = y^\alpha$, $\alpha \geq 1$ or $f(y) = \log(y)$ and ϕ is continuous.

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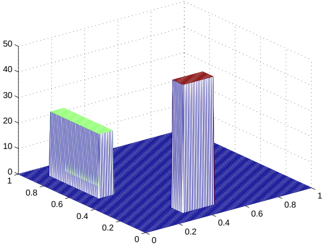
If agents have a *prior* ν on the other agents' actions, their cost-minimising behaviour leads to another *a posteriori* measure on the action space Y: $T\nu := (\mathrm{id} + \nabla\mathcal{V}[\nu])^{-1}_{\#}\mu$.
Assume

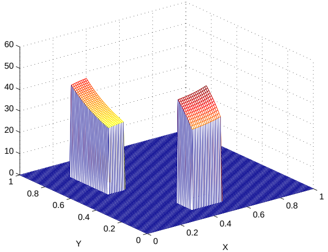
$$D^2\mathcal{V}[\nu_1] \geq \lambda\ \mathrm{id}\ , \quad \det(\mathrm{id} + D^2\mathcal{V}[\nu_1]) \leq M$$

$$\text{and} \quad \int_Y |\nabla\mathcal{V}[\nu_1](y) - \nabla\mathcal{V}[\nu_2](y)|\,\mathrm{d}y \leq C\mathcal{W}_1(\nu_1, \nu_2)\ .$$

THEOREM (CONVERGENCE OF THE BEST-REPLY ITERATION SCHEME, B.-CARLIER (2015))

If $M\,C\,\|\mu\|_{\mathrm{L}^\infty} < 1 + \lambda$ then for every $\nu_0 \in \mathcal{P}(Y)$, the sequence $(T^n\nu_0)_n$ converges to ν in the distance $\mathcal{W}_1$.





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CONNECTION WITH OPTIMAL TRANSPORT

Let $\gamma \in \mathcal{P}(X \times Y)$ be a Cournot-Nash equilibrium of second marginal ν. Then γ is a solution to the Kantorovich problem, *i.e.* γ is a solution to

$$\min_{\Pi_X \gamma = \mu, \Pi_Y \gamma = \nu} \iint_{X \times Y} c(x, y) \, \mathrm{d}\gamma(x, y) =: \mathcal{W}_c(\mu, \nu)$$

Proof: Let η be of first marginal μ and second marginal ν then we have

$$\begin{aligned} \iint_{X \times Y} c(x, y) \, \mathrm{d}\eta(x, y) &\geq \iint_{X \times Y} (\varphi(x) - \mathcal{V}[\nu](y)) \, \mathrm{d}\eta(x, y) \\ &= \int_X \varphi(x) \, \mathrm{d}\mu(x) - \int_Y \mathcal{V}[\nu](y) \, \mathrm{d}\nu(y) = \iint_{X \times Y} c(x, y) \, \mathrm{d}\gamma(x, y) \, . \end{aligned}$$

PURITY OF THE EQUILIBRIUM, BRENIER (1981)

If μ does not give weight to points then any Cournot-Nash equilibrium is pure.

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RESTRICTION: SYMMETRIC CASE

Assume that with $f(y) = \log(y)$ (or $f(y) = y^\alpha$, $\alpha \geq 1$) and $\phi(x,y) = \phi(|x-y|)$ with ϕ convex.

THEOREM (B., MOSSAY, SANTAMBROGIO, 2016 AND B.-CARLIER, 2015))

There exists a unique Cournot-Nash equilibrium.

Idea of the proof: Consider the free energy

$$\inf_{\nu \in \mathcal{P}(Y)} \left\{ \mathcal{W}_2(\mu,\nu)^2 + \underbrace{\int_Y \nu(y) \log \nu(y) \, \mathrm{d}y}_{\text{entropy}} + \underbrace{\frac{1}{2} \iint_{Y^2} \phi(|y-z|) \, \mathrm{d}\nu(y) \, \mathrm{d}\nu(z)}_{\text{interaction energy}} \right\} \tag{2}$$

+ perturb in the optimal transport sense.

EQUIVALENCE BETWEEN EQUILIBRIUM AND MINIMISER

$\gamma \in \mathcal{P}(X \times Y)$ is a Nash equilibrium if and only if

- ν is a minimiser of (2),
- γ is a solution to the Kantorovich problem.

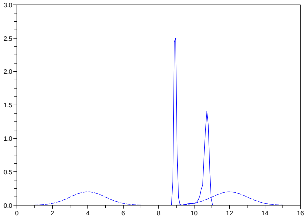
Idea of the proof: The above free energy (2) is convex along generalised geodesics.

A PARTIAL DIFFERENTIAL EQUATION FOR THE EQUILIBRIUM

Let u be a solution to the following Monge-Ampère equation

$$\mu(x) = \det(D^2 u(x)) \exp\left(-\frac{|\nabla u(x)|^2}{2} + x \cdot \nabla u(x) - u(x) - \int_Y \phi(\nabla u(y), \nabla u(z))\, \mathrm{d}\mu(z)\right)$$

then $\varphi(x) = u(x) + |x|^2/2$ is the optimal transport which transport μ onto ν so that $\nu = \varphi \# \mu$.



SOCIAL WELFARE

$$\iint_{X\times Y} F(x,y,\nu)\,\mathrm{d}\gamma = \iint_{X\times Y} \frac{|x-y|^2}{2}\,\mathrm{d}\gamma + \int_Y \left[\log(\nu(y)) + \int_Y \phi(|y-z|)\,\mathrm{d}\nu(z)\right]\,\mathrm{d}\nu(y).$$

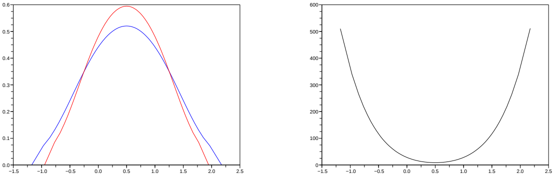


FIGURE : Left: the optimum (red) and the equilibrium (blue). Right: welfare transfer at the equilibrium. Cost of anarchy ~ 1.8.

WELFARE TRANSFER TO RESTORE EFFICIENCY

$$\mathrm{Transfer}[\nu](y) = \frac{1}{2}\int_Y \phi(|y-z|)\,\mathrm{d}\nu(z)\ .$$

The agents start with a distribution of strategies and adjust it over time by choosing

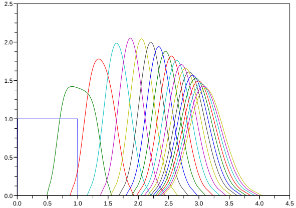
MINIMISING SCHEME

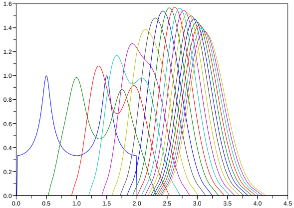
$$\nu_{k+1} \in \operatorname{argmin}_\nu \left\{ \frac{1}{2\tau} \mathcal{W}_2^2(\nu_k, \nu) + \mathcal{E}[\nu] \right\} .$$

This scheme converges in some sense to the

CONTINUOUS EVOLUTION EQUATION

$$\frac{\partial \nu}{\partial t} + \operatorname{div}\left(-\nu \nabla \mathcal{V}[\nu]\right) = 0,$$





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THEOREM (EXISTENCE OF EQUILIBRIA: NON-SYMMETRIC INTERACTION CASE (B.-CARLIER, 2014))

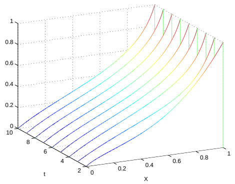
There exists at least one Cournot-Nash equilibrium.

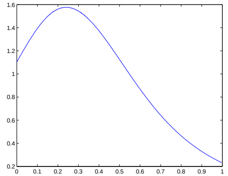
Idea of the proof: Consider

$$\inf_{\eta} \left\{ \mathcal{W}_2(\mu, \eta)^2 + \int_Y \log(\eta(y))\, \mathrm{d}y + \iint_{Y^2} \phi(y, z)\, \mathrm{d}\nu(y)\, \mathrm{d}\eta(z) \right\}$$

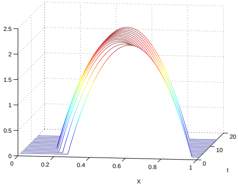
and perturb it in the L^2-sense + a fixed point argument
$\Rightarrow$ the optimality condition implies that γ is a Cournot-Nash equilibrium.

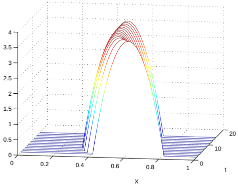
Logarithmic case:





Power case:





Merci beaucoup pour votre attention