Pontryagin Maximum Principle (PMP) for optimal nonpermanent control problems

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A simple optimal control problem

$$\text{minimize} \quad \int_0^T L(q(\tau), u(\tau), \tau)\, d\tau,$$

$$\text{subject to} \quad \begin{cases} \text{trajectory } q \in \mathrm{AC}([0,T],\mathbb{R}^n), & \\ \text{control } u \in \mathrm{L}^\infty([0,T],\Omega), & \Omega \subset \mathbb{R}^m \text{ nonempty}, \\ q'(t) = f(q(t), u(t), t), & \text{a.e } t \in [0,T], \\ q(0) = q_{\text{init}}, & q_{\text{init}} \in \mathbb{R}^n, \end{cases}$$

where $T > 0$ is fixed and where L and f are smooth enough.

Hamiltonian: $H(q,u,p,t) := \langle p, f(q,u,t) \rangle_{\mathbb{R}^n} - L(q,u,t)$.

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Historical PMP $\sim$ 1950's

If (q^*, u^*) is optimal, there exists an adjoint vector p such that $p(T) = 0$ and

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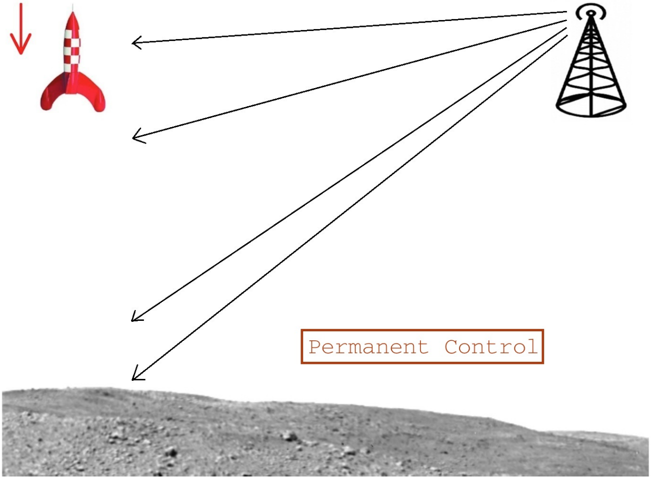
⚠ Classical control theories only deal with **permanent controls** ⚠



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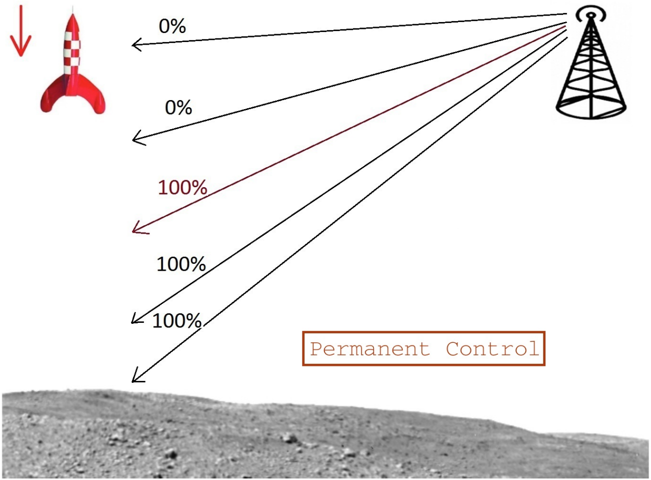

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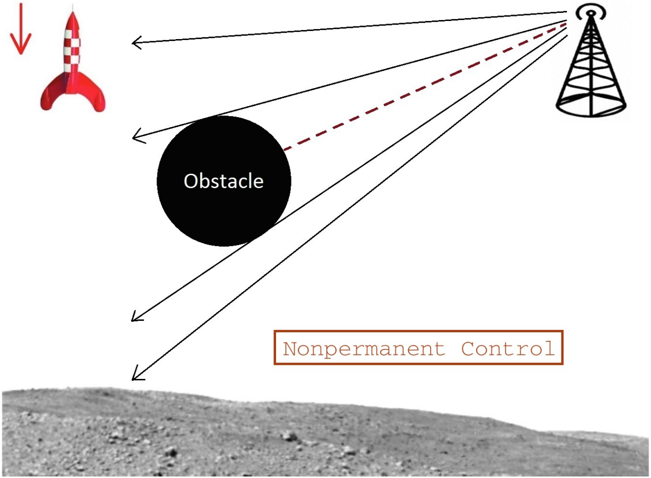

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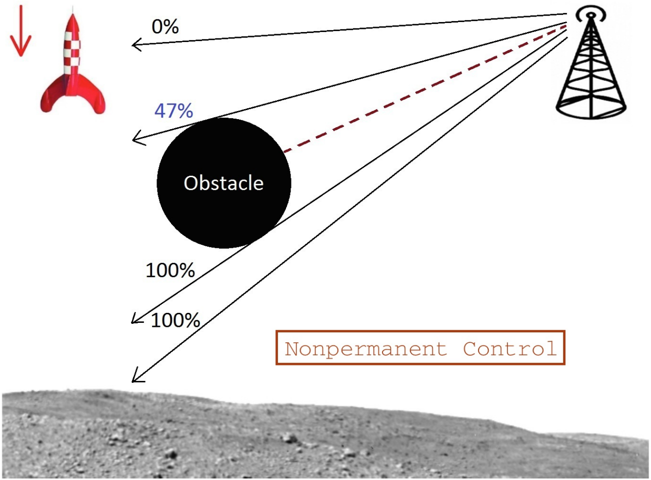
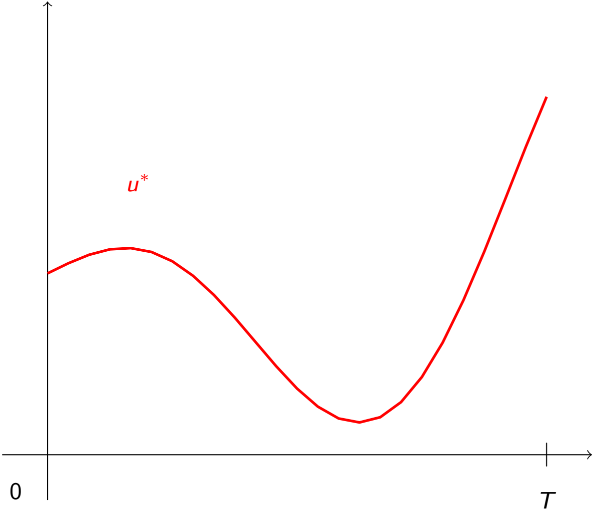


⚠ Optimal permanent controls may require a permanent ⚠ modification



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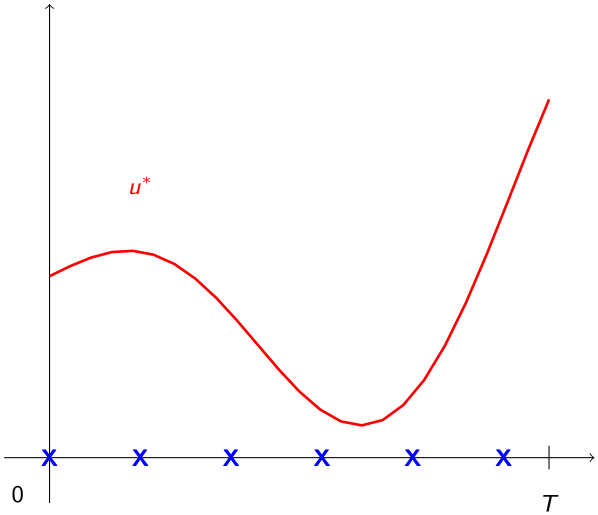

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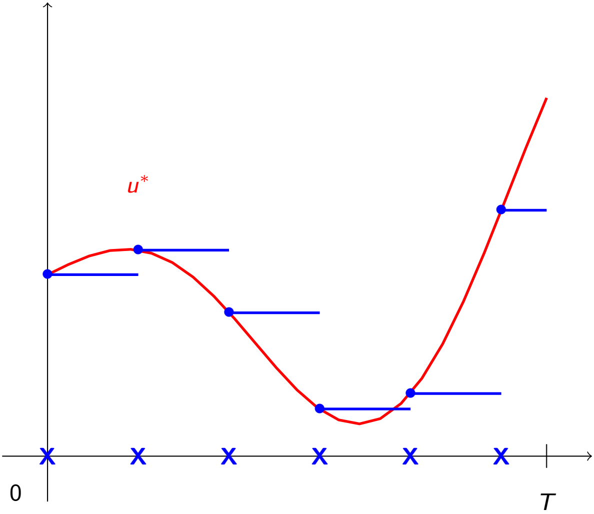
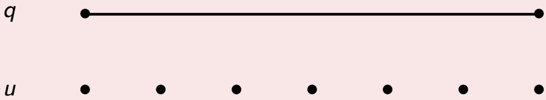


Permanent controls

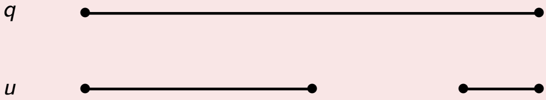
Permanent controls

Nonpermanent controls

Sampled-data controls:



Interval of non-control:



A simple optimal control problem (continuous case)

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Weak PMP (continuous case)

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A simple optimal control problem (discrete case)

$$\text{minimize} \quad \sum_{k=0}^{N-1} L(q_k, u_k, k),$$

$$\text{subject to} \quad \begin{cases} \text{discrete trajectory } q \in (\mathbb{R}^n)^N, & \\ \text{discrete control } u \in \Omega^N, & \Omega \subset \mathbb{R}^m \text{ nonempty convex}, \\ q_{k+1} - q_k = f(q_k, u_k, k), & \text{for all } k = 0, \dots, N-1, \\ q_0 = q_{\mathrm{init}}, & q_{\mathrm{init}} \in \mathbb{R}^n, \end{cases}$$

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PMP (discrete case) $\sim$ 1970's

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⚠ **Discrete case: no Hamiltonian maximization.** ⚠

Set of controlling times CT

In the sequel

- the trajectory q evolves continuously in time on $[0, T]$.
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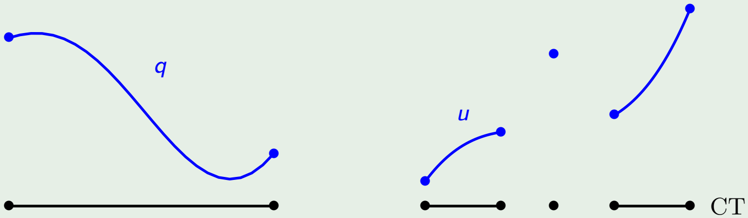
Nonpermanent controls and frozen values



Note that $\mathrm{CT} = \mathrm{RD} \cup \mathrm{RI}$ (**right-dense and right-isolated points**).

For $t \in \mathrm{RI}$, we denote by $\sigma(t) := \min\{s \in \mathrm{CT} \mid s > t\}$.

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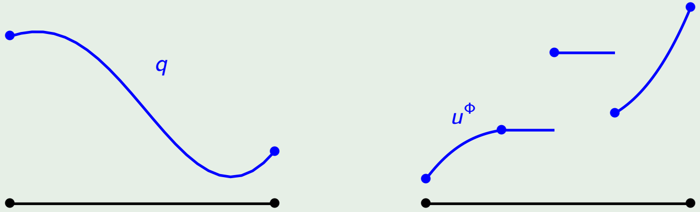

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Main result $\sim$ 2015

If (q^*, u^*) is optimal, there exists an adjoint vector p such that $p(T) = 0$ and

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and

- if $t \in \mathrm{RD}$: $\quad u^*(t) \in \arg\max_{v\in\Omega} H(q^*(t), v, p(t), t)$,
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Example: the linear-quadratic (LQ) case

$$\text{minimize} \quad \int_0^T q(\tau)^\top W(\tau) q(\tau) + u^\Phi(\tau)^\top R(\tau) u^\Phi(\tau) \, d\tau,$$

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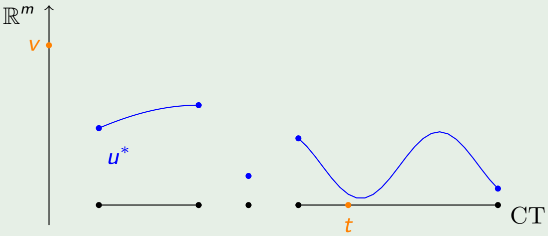
In the sampled-data control case $\mathrm{CT} = \{0 = t_0 < \ldots < t_N = T\}$, we obtain

$$u^*(t_k) = \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} R(\tau) \, d\tau \right)^{-1} \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} B(\tau) p(\tau) \, d\tau \right).$$

Needle-like variation of u^* at $t \in \mathrm{RD}$

$$\forall (t,v) \in \mathrm{RD}\times\Omega, \quad \forall \alpha > 0, \quad u_\alpha(\cdot) = \begin{cases} v & \text{on} \quad [t,t+\alpha)\cap \mathrm{CT}, \\ u^*(\cdot) & \text{elsewhere.} \end{cases}$$

Needle-like variation of type L^1 when $\alpha \to 0$



Variation vector ω

It holds that $q_\alpha = q^* + \alpha\,\omega + \text{rest}$ where

$$\begin{cases} \omega' = \partial_q f(q^*, u^{*\Phi}, \cdot) \times \omega, & \text{on } [t, T], \\ \omega(t) = f(q^*(t), v, t) - f(q^*(t), u^*(t), t). \end{cases}$$

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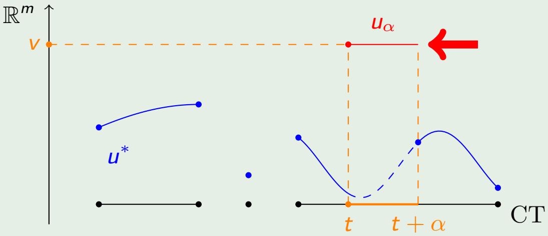

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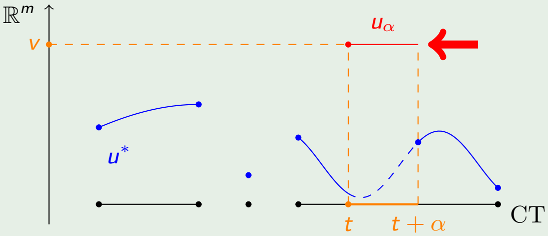

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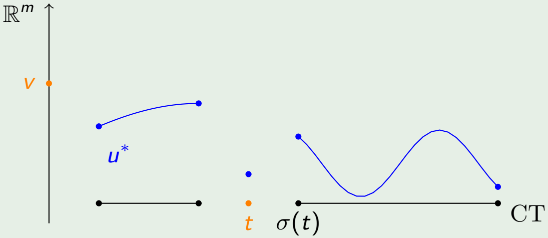

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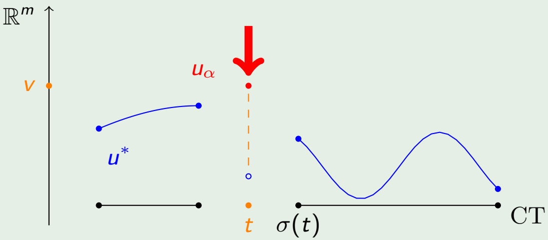

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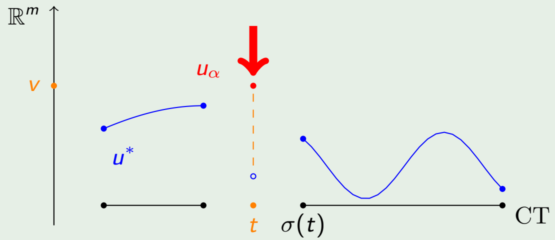

Variation vector ω

It holds that $q_\alpha = q^* + \alpha\, \omega + \text{rest}$ where

$$\begin{cases} \omega' = \begin{cases} \partial_q f(q^*, u^*(t), \cdot) \times \omega + \partial_u f(q^*, u^*(t), \cdot) \times (v - u^*(t)), & \text{on } [t, \sigma(t)], \\ \partial_q f(q^*, u^{*\Phi}, \cdot) \times \omega, & \text{on } [\sigma(t), T], \end{cases} \\ \omega(t) = 0. \end{cases}$$

Nonpermanent controls in the discrete setting?

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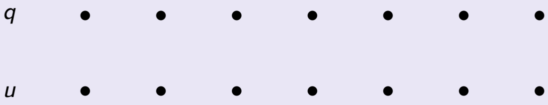
Permanent control in the discrete case



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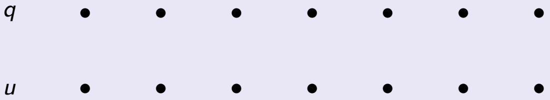

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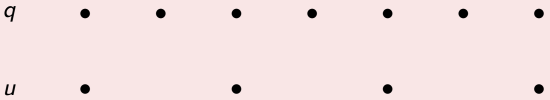

Time scale calculus theory

A **time scale** $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

The **time scale calculus theory** is based on notions of

- a Δ-measure on $\mathbb{T}$ (atomic measure);
- a Lebesgue Δ-integration theory for functions $q : \mathbb{T} \to \mathbb{R}^n$;
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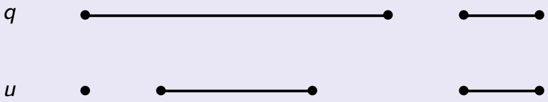
Time scale framework for nonpermanent controls

We fix two time scales $\mathbb{T}_1 \subset \mathbb{T} \subset \mathbb{R}$ and we consider that

- the trajectory q evolves on $\mathbb{T}$.
- the control u evolves on $\mathbb{T}_1$.



A more general framework

$$\text{minimize} \quad \psi(q(0), q(T), T) + \int_0^T L(q(\tau), u^{\Phi}(\tau), \tau)\, d\tau,$$

$$\text{s.t.} \begin{cases} \text{trajectory } q \in \mathrm{AC}([0,T], \mathbb{R}^n), & \\ \text{control } u \in \mathrm{L}^\infty(\mathrm{CT}, \Omega), & \Omega \subset \mathbb{R}^m \text{ nonempty closed}, \\ T \geq 0, & \\ q'(t) = f(q(t), u^{\Phi}(t), t), & \text{a.e. } t \in [0,T], \\ g(q(0), q(T)) \in \mathrm{S}, & \mathrm{S} \subset \mathbb{R}^j \text{ nonempty closed convex.} \end{cases}$$

- Calculus of variations $\rightsquigarrow$ Ekeland's variational principle.
  $\Rightarrow$ Closedness of Ω (reasonable assumption).
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- Ω non convex $\Rightarrow$ use of Ω-dense directions.

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Sufficient conditions

If the following hypotheses are satisfied:

- The set of admissible trajectories is not empty;
- There exists $M \geq 0$ such that
$$\|q\|_\infty + T \leq M$$
for all admissible trajectories;
- Ω is compact;
- For all $(q,t) \in \mathbb{R}^n \times [0,T]$, the set of extended velocities
$$\{(f(q,u,t), L(q,u,t)) \mid u \in \Omega\}$$
is convex;

then, there exists an optimal triplet (q^*, u^*, T^*).

We denote by $\mathcal{H}$ the application

$$\mathcal{H} : t \longmapsto H(q^*(t), {u^*}^\Phi(t), p(t), t).$$

Extension and nonextension of some classical properties

Continuous-Permanent case	**D-P case**	**C-NP case**
Hamiltonian maximization	×	×
$\mathcal{H}$ is continuous	✓	×
Autonomous case $\Rightarrow \mathcal{H}$ is constant	×	×
Free final time $T \Rightarrow \mathcal{H}(T^*) = 0$	×	✓
H affine in $u \Rightarrow$ saturation of Ω	✓	×

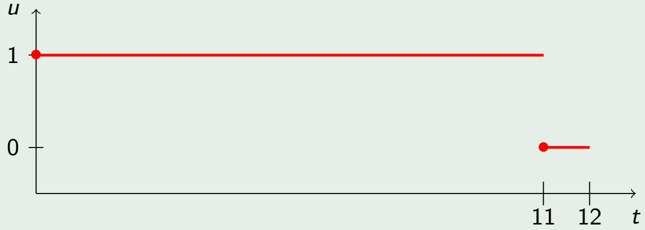
Linear example: optimal consumption

Consider the optimal (permanent) control problem given by

$$\text{maximize} \quad \int_0^{12} (1 - u(\tau))q(\tau)\, d\tau,$$

$$\text{subject to} \quad \begin{cases} q'(t) = u(t)q(t), & \text{a.e. on } [0,12], \\ q(0) = 1, \\ u(t) \in [0,1]. \end{cases}$$

The historical PMP gives the optimal (permanent) control:



Linear example: optimal consumption

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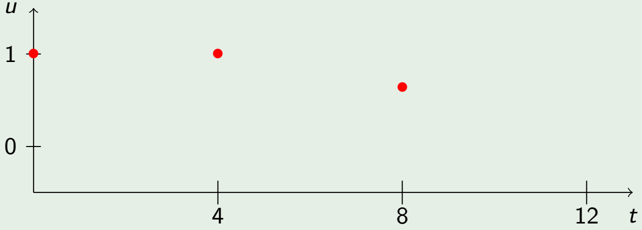
where $\mathrm{CT} = \{0, 4, 8\}$. Our main result provides



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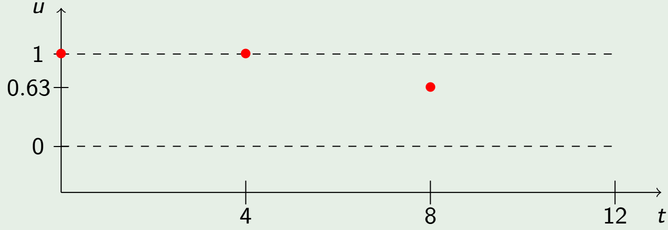

Sampled-data controls and optimal sampling times

$$\text{minimize} \quad \psi(q(0), q(T), T) + \int_0^T L(q(\tau), u^\Phi(\tau), \tau)\, d\tau,$$

$$\text{s.t.} \begin{cases} \text{trajectory } q \in \mathrm{AC}([0,T], \mathbb{R}^n), & \\ \text{control } u \in \mathrm{L}^\infty(\mathrm{CT}, \Omega), & \Omega \subset \mathbb{R}^m \text{ nonempty closed,} \\ \mathrm{CT} = \{0 < t_1 < \ldots < t_{N-1} < T\}, & \\ q'(t) = f(q(t), u^\Phi(t), t), & \text{a.e. } t \in [0,T], \\ g(q(0), q(T)) \in \mathrm{S}, & \mathrm{S} \subset \mathbb{R}^j \text{ nonempty closed convex,} \end{cases}$$

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We prove exactly the same PMP, with an additional necessary condition:

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$\mathcal{H}$ is continuous!

Remark: in the permanent case $\mathrm{CT} = [0, T]$, the set of controlling times is obviously optimal!

Example: the linear-quadratic (LQ) case

$$\text{minimize} \quad \int_0^T q(\tau)^\top W(\tau) q(\tau) + u^\Phi(\tau)^\top R(\tau) u^\Phi(\tau) \, d\tau,$$

$$\text{subject to} \begin{cases} \text{trajectory } q \in \mathrm{AC}([0,T],\mathbb{R}^n), \\ \text{control } u \in \mathrm{L}^\infty(\mathrm{CT},\mathbb{R}^m), \\ q'(t) = A(t)q(t) + B(t)u^\Phi(t), \quad \text{a.e } t \in [0,T], \\ q(0) = q_{\text{init}}, \qquad q_{\text{init}} \in \mathbb{R}^n. \end{cases}$$

where A, B, W and R are matrices with "good" assumptions.

In the permanent control case $\mathrm{CT} = [0,T]$, the historical PMP provides

$$u^*(t) = R(t)^{-1} B(t) p(t).$$

In the sampled-data control case $\mathrm{CT} = \{0 = t_0 < \ldots < t_N = T\}$, we obtain

$$u^*(t_k) = \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} R(\tau) \, d\tau \right)^{-1} \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} B(\tau) p(\tau) \, d\tau \right).$$

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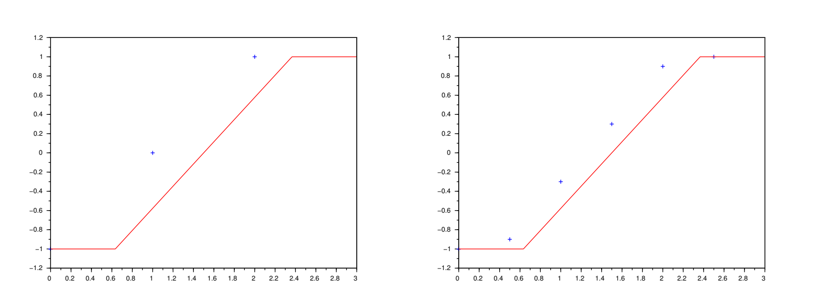
LQ example: the parking problem

Consider the optimal nonpermanent control problem given by

$$\text{minimize} \quad \int_0^T u^\Phi(\tau)^2 \, d\tau,$$

$$\text{subject to} \quad \begin{cases} q''(t) = u^\Phi(t), \\ (q(0), q'(0)) = (M, 0), \ (q(T), q'(T)) = (0, 0), \\ u(t) \in [-1, 1], \end{cases}$$

where $\mathrm{CT} = h\mathbb{N}$. We obtain for $h = 1$ and $h = 0.5$:



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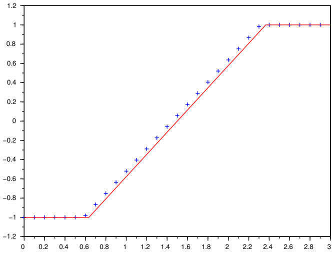

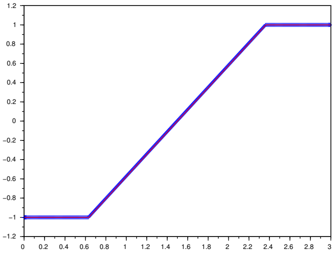

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You can access to these articles on my personal web page

`www.unilim.fr/pages_perso/loic.bourdin`

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Future works

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THANK
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