L^1-Optimality Conditions

for the Circular Restricted Three-Body Problem

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adjoint work with

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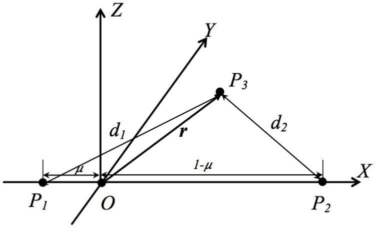
Circular restricted three-body problem (CRTBP)

The CRTBP consists of three gravitationally interacting bodies, P_1, P_2, and P_3, whose masses are denoted by m_1, m_2, and m_3, respectively, such that

1. the third mass m_3 is so small that its gravitational influence on the other two is negligible;
2. the two primaries, P_1 and P_2, move on circular orbits around their common centre of mass.

The length is normalized by $d_* > 0$, the distance between P_1 and P_2.

If $\mu = m_2/(m_1+m_2)$, $r_1 = (-\mu, 0, 0)$ and $r_2 = (1-\mu, 0, 0)$ denote the position of P_1 and P_2, respectively.



Rotating frame for the CRTBP

Dynamics of the CRTBP

$r \in \mathbb{R}^3$:= position vector, $v \in \mathbb{R}^3$:=velocity vector,
$m \in \mathbb{R}_+$:= mass , $\mathscr{X} \subset \mathbb{R}^n$:= the admissible set of $x = (r, v, m)$.

The controlled equation for the CRTBP is

$$\Sigma : \begin{cases} \dot{r}(t) = v(t), \\ \dot{v}(t) = h(v(t)) + g(r(t)) + \frac{\tau(t)}{m(t)}, \\ \dot{m}(t) = -\beta \|\tau(t)\|, \end{cases}$$

$$h(v) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v, \quad g(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} r - \frac{1-\mu}{\|r - r_1\|^3}(r - r_1) - \frac{\mu}{\|r - r_2\|^3}(r - r_2).$$

$\beta \geq 0$ is a constant and $\tau \in \mathbb{R}^3$ is the thrust (or control) vector valued in a Euclidean ball

$$\mathscr{B}_\tau = \{\tau \in \mathbb{R}^3 \mid \|\tau\| \leq \tau_{\max}\}, \quad \tau_{\max} \text{ is a positive constant.}$$

Dynamics of the CRTBP

Let $(\rho, \omega) \in [0,1] \times \mathbb{S}^2$ such that

$$\rho = \| \tau \| / \tau_{max}, \quad \tau = \rho \tau_{\max} \omega.$$

We rewrite the system Σ as

$$\Sigma : \dot{x}(t) = f(x(t), \rho(t), \omega(t)) = f_0(x(t)) + \rho(t) f_1(x(t), \omega(t)),$$

where

$$f_0(\mathbf{x}) = \begin{pmatrix} v \\ h(v) + g(r) \\ 0 \end{pmatrix}, \quad f_1(x, \omega) = \begin{pmatrix} 0 \\ \frac{\tau_{\max}}{m} \omega \\ -\beta \tau_{\max} \end{pmatrix}.$$

L^1-Minimization

Define the constraint submanifold of target by

$$\mathscr{M} = \{\boldsymbol{x} \in \mathscr{X} \mid \phi(\boldsymbol{x}) = 0\},$$

where $\phi : \mathscr{X} \to \mathbb{R}^l$ is twice continuously differentiable.

L^1-minimization

$$\dot{x}(t) = f(x(t), \rho(t), \omega(t)), \qquad x(t) \in \mathscr{X} \subset \mathbb{R}^n, \qquad (\rho(t), \omega(t)) \in \mathscr{U},$$

$$x(0) = x_0, \qquad x(t_f) \in \mathscr{M}, \qquad t_f > 0 \text{ is fixed},$$

$$\int_0^{t_f} \rho(t)dt \to \min.$$

Pontryagin Maximum Principle (PMP)

Pontragin maximum prinicple

Let $u = (\rho, \omega)$. Every minimizing trajectory $x(\cdot)$ is the projection of an extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ solution of

$$\dot{x}(t) = \frac{\partial H}{\partial p}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}, \quad H(x, p, p^0, u) = \max_{\eta \in \mathscr{U}} H(x, p, p^0, \eta),$$

where $H(x, p, p^0, u) = \langle p, f(x, u) \rangle + p^0 \rho$.

An extremal is said normal if $p^0 \neq 0$, and abnormal if $p^0 = 0$ (abnormal extremals have been ruled out by Caillau *et al.* (2012)).

In the normal case, the maximum Hamiltonian can be written as

$$H(x, p) = H_0(x, p) + \rho(x, p) H_1(x, p),$$

where $H_0 = \langle p, f_0(x) \rangle$ and $H_1 = \langle p, f_1(x, \omega(x, p)) \rangle - 1$.

Bang-bang & Singular controls

Let $p = (p_r, p_v, p_m)$ such that

$$H = \underbrace{\langle p_r, v\rangle + \langle p_v, h(v) + g(r)\rangle}_{H_0} + \rho\big(\underbrace{\langle p_v, \omega/m\rangle - \beta p_m}_{H_1}\big).$$

The maximum condition implies

$$\omega = p_v/\|p_v\|, \text{ if } \|p_v\| \neq 0,$$

and

$$\rho = \begin{cases} 1, \text{ if } H_1 > 0, \\ 0, \text{ if } H_1 < 0, \end{cases} \quad \Longrightarrow \quad \text{norm of control is bang-bang}.$$

If H_1 has only isolated zeros on $[0, t_f]$, the corresponding extremal is called a nonsingular one; If $H_1 \equiv 0$ on $[0, t_f]$, the corresponding extremal is called a singular one.

Singular extremals and chattering phenomena

Kelley (1962)

ρ appears in $\frac{d^q H_1}{dt^q}$ if q is even, and $q/2$ is the order of the singular extremals.

The order of a singular extremal $(x(\cdot), p(\cdot))$ on $[t_1, t_2] \subseteq [0, t_f]$ with $t_1 < t_2$ is two, i.e., $\frac{d^4 H_1}{dt^4} = \alpha\rho + \beta$ with $\alpha \neq 0$. Kelley's second order necessary condition is $\alpha \leq 0$.

$$\mathscr{S} = \{(x,p) \in T^*\mathscr{X} \mid H_1 = \frac{dH_1}{dt} = \frac{d^2 H_1}{dt^2} = \frac{d^3 H_1}{dt^3} = \alpha\rho + \beta = 0,\ \alpha \leq 0\}.$$

Theorem (Zelikin and Borisov, 1994 & 2003)

Let $int(\mathscr{S})$ be the interior of $\mathscr{S}$. Then, given every point $(x,p) \in int(\mathscr{S})$, there exists a one parameter family of chattering solutions to the PMP passing through the point (x,p) and another one parameter family of chattering solutions to the PMP coming out from the point $(\boldsymbol{x}, \boldsymbol{p})$.

Sufficient Conditions for Optimality

Definition of local optimality

Local optimality

Given a fixed final time $t_f > 0$, an extremal trajectory $\bar{\boldsymbol{x}}(\cdot) \in \mathscr{X}$ associated with the extremal control $\bar{\boldsymbol{u}}(\cdot) = (\bar{\rho}(\cdot), \bar{\omega}(\cdot))$ in $\mathscr{U}$ on $[0, t_f]$ is said to realize a strong-local optimality in C^0-topology if there exists an open neighborhood $\mathscr{W}_{\boldsymbol{x}} \subseteq \mathscr{X}$ of $\bar{\boldsymbol{x}}(\cdot)$ in C^0-topology such that for every admissible controlled trajectory $\boldsymbol{x}(\cdot) \not\equiv \bar{\boldsymbol{x}}(\cdot)$ in $\mathscr{W}_{\boldsymbol{x}}$ associated with the measurable control $\boldsymbol{u}(\cdot) = (\rho(\cdot), \omega(\cdot))$ in $\mathscr{U}$ on $[0, t_f]$ with the boundary conditions $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}(0)$ and $\boldsymbol{x}(t_f) \in \mathscr{M}$, there holds

$$\int_0^{t_f} \rho(t)dt \geq \int_0^{t_f} \bar{\rho}(t)dt.$$

We say it realizes a strict strong-local optimality if the strict inequality holds.

Parameterized family of extremals

Parameterized family of extremals

Given the reference extremal $(\bar{x}(\cdot), \bar{p}(\cdot))$ on $[0, t_f]$, let $\mathscr{P} \subset T^*_{x_0}\mathscr{X}$ be an open neighborhood of $\bar{p}(0)$, we say the subset

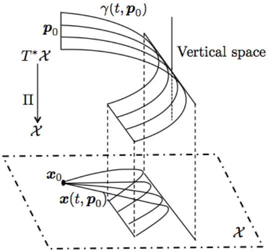
$$\mathscr{F} = \{(x(t), p(t)) \in T^*\mathscr{X} \mid (x(t), p(t)) = e^{t\vec{H}}(\bar{x}(0), p_0),\ t \in [0, t_f],\ p_0 \in \mathscr{P}\},$$

a p_0-parameterized family of extremals around the reference one.

$$\Pi : T^*\mathscr{X} \to \mathscr{X},\ (\boldsymbol{x}, \boldsymbol{p}) \mapsto \boldsymbol{x}.$$

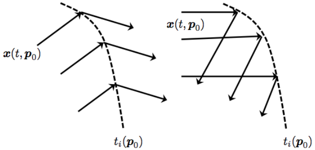
Conjugate point and fold singularity

Smooth Fold Singularity



Agrachev, A. A.; Sachkov, Y. L. (2004)

Broken Fold Singularity



Schattler, H.; Noble, J. (2012)

No-fold conditions

Let $(x(\cdot,p_0),p(\cdot,p_0)) := e^{t\vec{H}}(\bar{x}(0),p_0)$ on $[0,t_f]$ and let $\delta(t) := \det\left[\frac{\partial x}{\partial p_0}(t,\bar{p}_0)\right]$.

No-fold condition on smooth bang arcs

$\delta(\cdot) \neq 0$ on (t_i, t_{i+1}). (t_i is the switching time, i.e., $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = t_f$.)

No-fold condition at switching times

$\delta(t_i-)\delta(t_i+) > 0$.

The no-fold conditions were established in

"L^1-Minimization for Mechanical Systems" to appear in SIAM Journal on Control and Optimization (with J.-B. Caillau and Y. Chitour).

Sufficient conditions for $l = n$

Perturbed Lagrangian submanifold $\mathscr{L}$ (Agrachev *et al.* (2004))

If $(x(\cdot,\bar{p}_0), p(\cdot,\bar{p}_0))$ on $(0, t_f]$ does not contain conjugate points, we are able to construct a perturbed Lagrangian submanifold $\mathscr{L} \in T^*\mathscr{X}$ such that

1. the projection Π of $\mathscr{L}$ onto its image is a local diffeomorphism; and
2. the domain $\Pi(\mathscr{L})$ is an open neighborhood of the extremal trajectory $\boldsymbol{x}(\cdot,\bar{p}_0) = \Pi(x(\cdot,\bar{p}_0), p(\cdot,\bar{p}_0))$ on $[0, t_f]$ in C^0-topology.

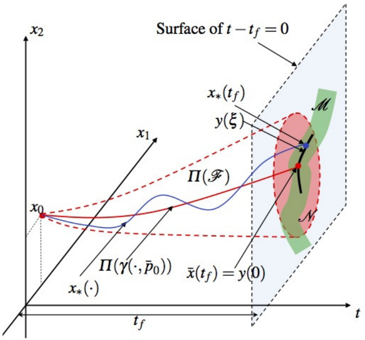
Theorem (Agrachev *et al.*, 2004)

Given a bang-bang extremal $(\bar{x}(t), \bar{p}(t))$ on $[0, t_f]$, if $\delta(\cdot) \neq 0$ on (t_i, t_{i+1}) and if $\delta(t_i-)\delta(t_i+) > 0$, the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$ realizes a strict minimum cost among all the admissible controlled trajectories $x(\cdot)$ on $[0, t_f]$ in the domain $\Pi(\mathscr{L})$ with the same endpoints: $\bar{x}(0) = x(0)$ and $\bar{x}(t_f) = x(t_f)$.

Recall that the Poincaré-Cartan form $pdx - Hdt$ is exact on $\mathscr{L}$.

Sufficient conditions for $l < n$

If the final point is not fixed but varies on $\mathscr{M}$, in addition to the two no-fold conditions, an extra second-order condition is required to guarantee that every admissible controlled trajectory $x_*(\cdot) \in \mathscr{W}_x$ on $[0, t_f]$ with the boundary conditions $x_0 = x_*(0)$ and $x_*(t_f) \in \mathscr{M} \backslash \{\bar{x}(t_f)\}$ has a higher cost than the reference one.



$$\gamma(t, p_0) = e^{t\vec{H}}(\bar{x}(0), p_0)$$

Sufficient conditions for $l < n$

Define $\bar{v} \in (\mathbb{R}^l)^*$ such that $\bar{p}(t_f) = \bar{v} d\phi(\bar{x}(t_f))$.

$$\zeta^T \left\{ \frac{\partial p}{\partial p_0}(t_f, \bar{p}_0) \left[\frac{\partial x}{\partial p_0}(t_f, \bar{p}_0) \right]^{-1} - \bar{v} d^2\phi(\bar{x}(t_f)) \right\} \zeta > 0 \text{ for every } \zeta \in T_{\bar{x}(t_f)}\mathscr{M}.$$

This conditions was established in

Z. Chen, "L^1-optimality conditions for the circular restricted three-body problem", arXiv, 2015.

Theorem

In the case of $l < n$, given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot))$ on $[0, t_f]$ such that the no-fold conditions are satisfied, the reference extremal realizes a strong local optimum if there holds

$$\frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v} d^2\phi(\bar{x}(t_f)) \succ 0,$$

on the tangent space $T_{\bar{x}(t_f)}\mathscr{M}$.

Numerical implementation

Differential equations:

$$\begin{bmatrix} \frac{d}{dt}\frac{\partial x}{\partial p_0}(t,\bar{p}_0) \\ \frac{d}{dt}\frac{\partial p^T}{\partial p_0}(t,\bar{p}_0) \end{bmatrix} = \begin{bmatrix} H_{px} & H_{pp} \\ -H_{xx} & -H_{xp} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial p_0}(t,\bar{p}_0) \\ \frac{\partial p^T}{\partial p_0}(t,\bar{p}_0) \end{bmatrix}.$$

Jacobi fields jump at t_i:

$$\frac{\partial x}{\partial p_0}(t_i+,\bar{p}_0) = \frac{\partial x}{\partial p_0}(t_i-,\bar{p}_0) - \Delta\rho_i \frac{\partial H_1}{\partial p}\frac{dt_i(\bar{p}_0)}{dp_0}.$$

$$\frac{\partial p^T}{\partial p_0}(t_i+,\bar{p}_0) = \frac{\partial p^T}{\partial p_0}(t_i-,\bar{p}_0) + \Delta\rho_i \frac{\partial H_1}{\partial x}\frac{dt_i(\bar{p}_0)}{dp_0}$$

Initial condition:

$$\frac{\partial x}{\partial p_0}(0,\bar{p}_0) = 0 \text{ and } \frac{\partial p^T}{\partial p_0}(0,\bar{p}_0) = I_n.$$

Fuel-optimal problem with variable endpoints

Denote the boundary constraint manifolds by

$$\mathscr{M}_i = \{x \in \mathscr{X} \mid \phi_i(x) = 0\} \text{ and } \mathscr{M}_f = \{x \in \mathscr{X} \mid \phi_f(x) = 0\},$$

where $\phi_i : \mathscr{X} \to \mathbb{R}^{l_i}$ $(0 < l_i < n)$ and $\phi_f : \mathscr{X} \to \mathbb{R}^{l_f}$ $(0 < l_f < n)$.

Fuel-optimal problem with variable endpoints

$$\beta > 0, \quad t_f > 0, \quad x(0) \in \mathscr{M}_i, \quad x(t_f)\mathscr{M}_f,$$

$$\dot{x}(t) = f_0(x(t)) + f_1(x(t), \omega(t)), \quad (\rho(t), \omega(t)) \in [0,1] \times \mathbb{S}^2,$$

$$\int_0^{t_f} \rho(t)dt \to \min.$$

$$\int_0^{t_f} \rho(t)dt \to \min \implies m(t_f) \to \max.$$

Test sufficient conditions backward

Why to test sufficient conditions backward for fuel-optimal problem?

For fuel-optimal problem, m is a state instead of a constant parameter.

$$\rho \text{ is a piece-wise constant} \Longrightarrow \quad \frac{d}{dt}\frac{\partial m}{\partial p_0}(\cdot,\bar{p}_0) \equiv 0 \text{ on } (t_i, t_{i+1}).$$
$$m(0) \text{ is fixed} \Longrightarrow \quad \frac{\partial m}{\partial p_0}(0,\bar{p}_0) = 0.$$

It is concluded that $\det\left[\frac{\partial x}{\partial p_0}(t,\bar{p}_0)\right] = 0$ on $[0, t_1]$.

Parameterized family of extremals

Define the Lagrangian submanifold

$$\mathscr{L}_f := \{(x,p) \in T^*\mathscr{X} \mid x \in \mathscr{M}_f,\ p \perp T_x\mathscr{M}_f\}.$$

Locally, there exists a diffeomorphism $F : \mathscr{L}_f \to (\mathbb{R}^n)^*$ such that for every $(x,p) \in \mathscr{L}_f$ there exists one and only one $q \in (\mathbb{R}^n)^*$ with $F(x,p) = q$.

Parameterized family of extremals

Let $\bar{q} := F^{-1}(\bar{x}(t_f), \bar{p}(t_f))$. Given the reference extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = e^{(t-t_f)\vec{H}}(F^{-1}(\bar{\mathbf{q}}))$ on $[0, t_f]$, let $\mathscr{Q} \subset \mathscr{L}_f$ be a sufficiently small open neighborhood of $\bar{q}$, we say the subset

$$\mathscr{F}_q = \{(x(t), p(t)) \in T^*\mathscr{X} \mid (x(t), p(t)) = e^{(t-t_f)\vec{H}}(F^{-1}(q)),\ t \in [0, t_f],\ q \in F(\mathscr{Q})\},$$

a q-parameterized family of extremals around the reference one.

Sufficient conditions for fuel-optimal problem

Let $(x(t,q),p(t,q)) := e^{(t-t_f)\vec{H}}(F^{-1}(q))$ and $\delta_q(t) = \det\left[\frac{\partial x}{\partial q}(t,\bar{q})\right]$.

No-fold condition on smooth bang arcs

$\delta_q(\cdot) \neq 0$ on (t_i, t_{i+1}).

No-fold condition at switching times

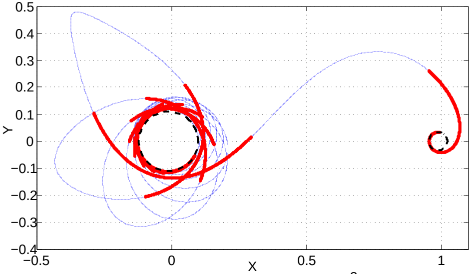
$\delta_q(t_i-)\delta_q(t_i+) > 0$.

The third condition

$\zeta^T\left\{\frac{\partial p}{\partial q}(0,\bar{q})\left[\frac{\partial x}{\partial q}(0,\bar{q})\right]^{-1} - \nu_i d^2\phi_i(\bar{x}(0))\right\}\zeta < 0$ for every $\zeta \in T_{\bar{x}(0)}\mathscr{M}_i$.

Numerical Applications

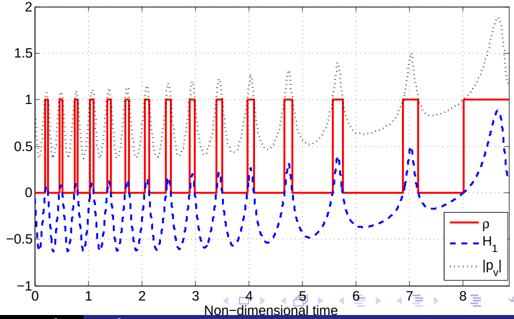
Conjugate point test: CRTBP (mass constant model with variable target)



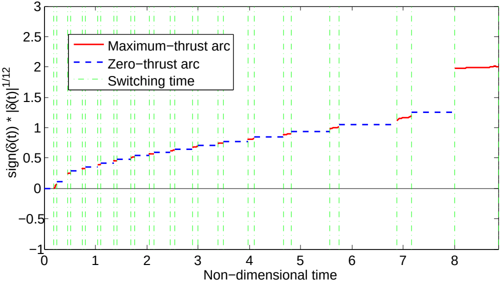
Boundary conditions

Initial state $x(0)$ is fixed and final state $x(t_f)$ varies on a circular orbit of the Moon.

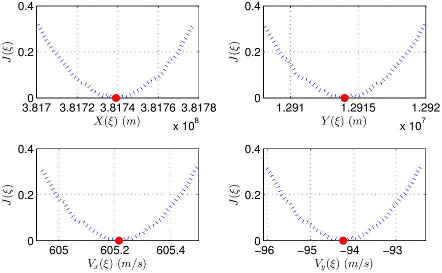
$\mu :=$	0.01215
$\tau_{\max} :=$	1.0 N
$m_0 :=$	500 kg
$\beta :=$	0



Conjugate point test: CRTBP (mass constant model with variable target)

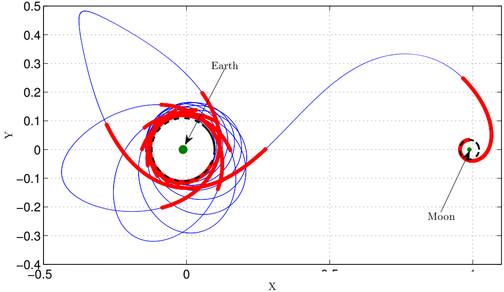


$\delta(\cdot) \neq 0$ on $(0, t_f]$ and $\delta(t_i-)\delta(t_i+) > 0$.



$$C^T \left\{ \frac{\partial p^T(t_f,\bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f,\bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v} d^2\phi(\bar{x}(t_f)) \right\} C \approx 0.529 \quad > \quad 0.$$

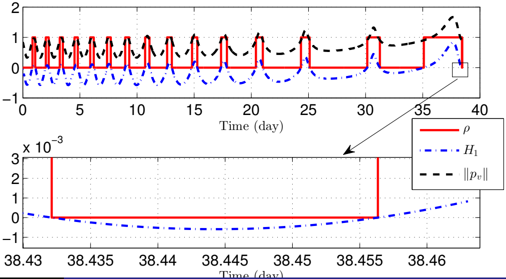
Focal point test: case A (mass varying model with fixed initial point)



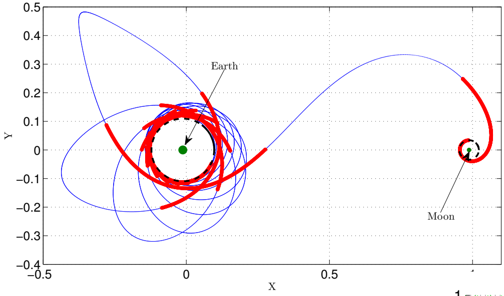
Boundary conditions

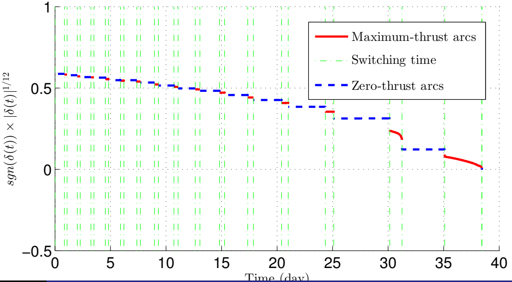
Initial state is fixed, while the final point varies on a circular orbit of the Moon.

μ	:=	0.01215
$\tau_{\max}$	:=	1.0 N
m_0	:=	500 kg
I_{sp}	:=	2000 s
g_0	:=	9.81 m/s^2
β	:=	$1/(I_{sp}g_0) > 0$



Focal point test: case A (mass varying model with fixed initial point)

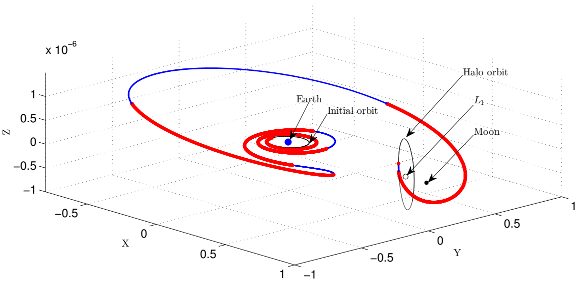




$\delta_q(\cdot) \neq 0$ on $[0, t_f)$ and $\delta_q(t_i^-)\delta_q(t_i+) > 0$, ensuring the optimum of the computed trajectory.

Focal point test: case B (mass varying model with variable initial point)

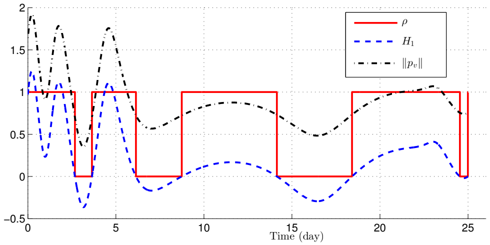
——a non-optimal example



Boundary conditions

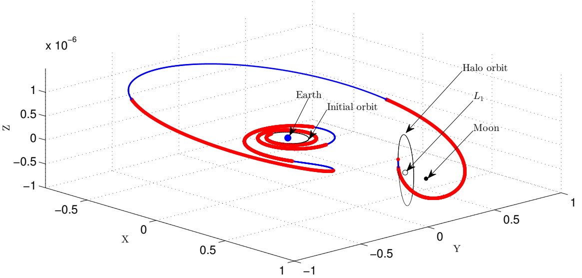
Initial point varies on a circular orbit of the Earth, while the final point is fixed on a Halo orbit with final mass free.

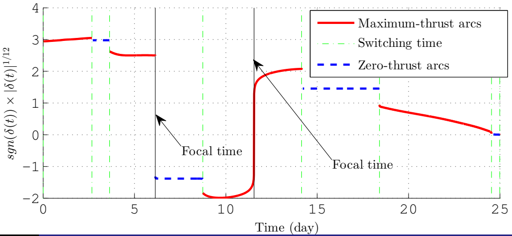
μ	:=	0.01215
$\tau_{\max}$	:=	1.0 N
m_0	:=	300 kg
I_{sp}	:=	2000 s
g_0	:=	$9.81\ m/s^2$
β	:=	$1/(I_{sp}g_0) > 0$



Focal point test: case B (mass varying model with variable initial point)

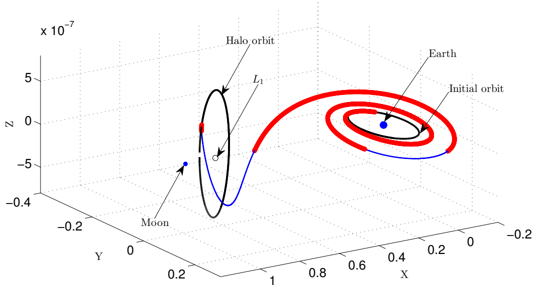
——a non-optimal example





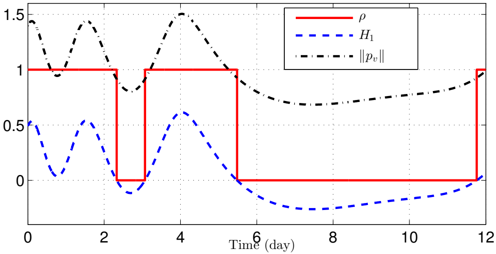
One focal point occurs at a switching time and another one occurs on a burn arc.

Focal point test: case C (the same boundary conditions as case B)

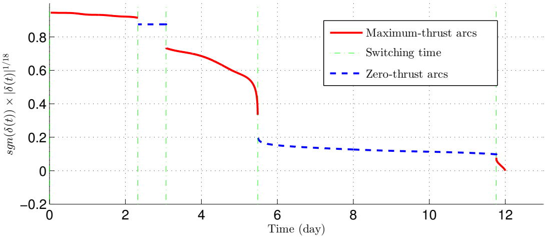


The boundary conditions are the same as those of case B, but the transfer time is specified to 12 days.

$\mu :=$	0.01215
$\tau_{\max} :=$	1.0 N
$m_0 :=$	300 kg
$I_{sp} :=$	2000 s
$g_0 :=$	9.81 m/s^2
$\beta :=$	$1/(I_{sp} g_0) > 0$

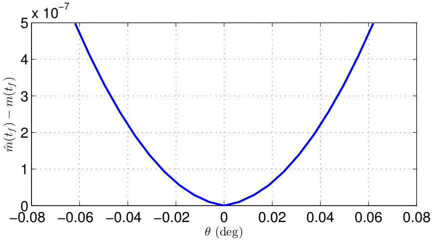


Focal point test: case C (the same boundary conditions as case B)



$\delta_q(\cdot) \neq 0$ on (t_i, t_{i+1}) and $\delta_q(t_i-)\delta_q(t_i+) > 0$, ensuring the optimum of the computed trajectory.

$$C^T\left\{\frac{\partial p^T(0,\bar{q})}{\partial q}\left[\frac{\partial x(0,\bar{q})}{\partial q}\right]^{-1} - \bar{\nu}_i d^2\phi_i(\bar{x}(0))\right\}C \approx -3.082\times 10^{-2} < 0.$$



Thank you for your attention!

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