Earth-Moon Mission between Libration Points: Low Thrust Transfer

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Journées SMAI-MODE 2016

25th March, 2016

Outline

- Three Body Problem
  - Lyapunov/Halo Orbits
  - Invariant Manifolds
- Lyapunov to Lyapunov Mission
  - Optimal Control Problem
  - Using Invariant Manifolds

Circular Restricted 3 Body Problem

- A satellite P with negligible mass m.
- 2 primaries P_1 and P_2 rotating around their center of mass.
- Respective mass: M_1 and M_2.

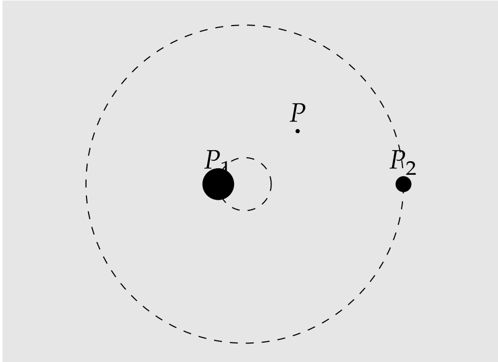
$$m\frac{\mathrm{d}\underline{R}}{\mathrm{d}t} = -GM_1 m\frac{\underline{R}_{13}}{R_{13}^3} - GM_2 m\frac{\underline{R}_{23}}{R_{23}^3}$$

- Normalize the system:

$$\begin{aligned} &\text{distance} && d' = l_* d, \\ &\text{velocity} && s' = v_* s, \\ &\text{time} && t' = \frac{t_*}{2\pi} t. \end{aligned}$$

- The mass parameter:

$$\mu = \frac{M_2}{M_1 + M_2}.$$



- Rotating coordinate system in which the two primaries are fixed (along the x axis).
- In this work Earth-Moon system.

Normalized System

- State:

$$\chi = (x, y, \dot{x}, \dot{y})^T = (x_1, x_2, x_3, x_4)^T,$$

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_1 + 2x_4 - (1-\mu)\dfrac{x_1 - x_1^0}{r_1^3} - \mu\dfrac{x_1 - x_2^0}{r_2^3} \\ \dot{x}_4 = x_2 - 2x_3 - (1-\mu)\dfrac{x_2}{r_1^3} - \mu\dfrac{x_2}{r_2^3} \end{cases} \Leftrightarrow \begin{cases} \dot{x}_1 = x_3 = f_1(\chi) \\ \dot{x}_2 = x_4 = f_2(\chi) \\ \dot{x}_3 = 2x_4 - \dfrac{\partial U}{\partial x_1} = f_3(\chi) \\ \dot{x}_4 = -2x_3 - \dfrac{\partial U}{\partial x_2} = f_4(\chi) \end{cases}$$

where

$$r_1 = \sqrt{\left(x_1 - x_1^0\right)^2 + x_2^2} \quad \text{and} \quad r_2 = \sqrt{\left(x_1 - x_2^0\right)^2 + x_2^2}$$

are respectively the distances between P and primaries P_1 and P_2.

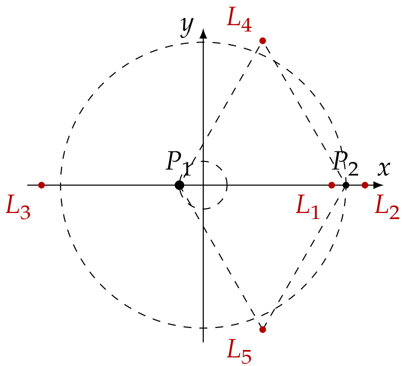
- Potential:

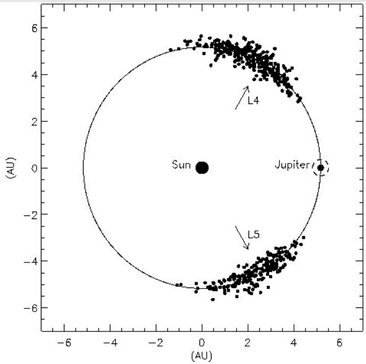
$$U(x_1, x_2) = -\frac{1}{2}\left(x_1^2 + x_2^2\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1-\mu).$$

- $\dot{\chi} = F_0(\chi)$.

Lagrange Points

- Five equilibrium points called Lagrange points.
- Collinear points: L_1, L_2 and L_3.
- Equilateral points: L_4 and L_5.

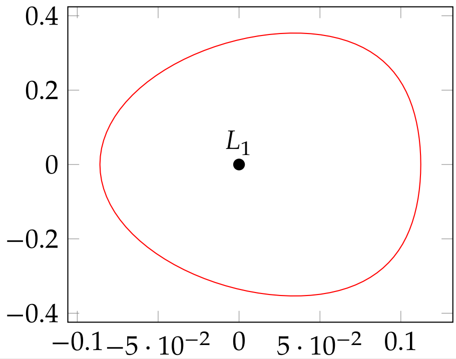




- Stability is studied by linearization.
- L_1, L_2 and L_3 are instable.
- Stability of L_4 and L_5 depends on the system.

Lyapunov Orbits

- The Lyapunov-Poincaré theorem ensures that a family of periodic orbits exists around Lagrange points.
- Method based on symmetry.
- Newton method.
- Initialization with analytical approximation (Richardson 80).



- There exist t_0 and t_χ s.t.

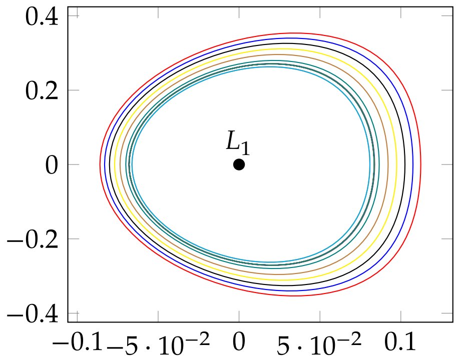
$$\begin{cases} x(t_0) = x_0, \\ y(t_0) = 0, \\ \dot{x}(t_0) = 0, \\ \dot{y}(t_0) = \dot{y}_0, \end{cases} \text{and} \begin{cases} x(t_0 + t_\chi/2) = x_1, \\ y(t_0 + t_\chi/2) = 0, \\ \dot{x}(t_0 + t_\chi/2) = 0, \\ \dot{y}(t_0 + t_\chi/2) = \dot{y}_1. \end{cases}$$

- Taking $t_0 = 0$, and fixing x_0, we define the shooting function:

$$\Longrightarrow \mathcal{S}_L(t_\chi, \dot{y}_0) = \begin{pmatrix} \phi_2(t_\chi/2, \chi_0) \\ \phi_3(t_\chi/2, \chi_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{where } \chi_0 = (x_0, 0, 0, \dot{y}_0).$$

Family of Lyapunov Orbits

- Continuation method: define a family of problems $\mathcal{P}_\lambda$, such that $\mathcal{P}_0$ is easy to solve, and $\mathcal{P}_1$ is the problem we want to solve.
- Initialization of $\mathcal{P}_{\lambda_i}$ with solution of $\mathcal{P}_{\lambda_{i-1}}$.
- We have solved the problem for a given $\mathcal{E}_0$.
- We want to reach a certain energy $\mathcal{E}_1$.



- Continuation problems:

We define $\mathcal{E}(\chi_0)$ as the energy of the trajectory starting at $\chi_0 = (x_0, 0, 0, \dot{y}_0)$ and

$$\mathcal{E}_\lambda = (1-\lambda)\mathcal{E}_0 + \lambda\mathcal{E}_1,$$

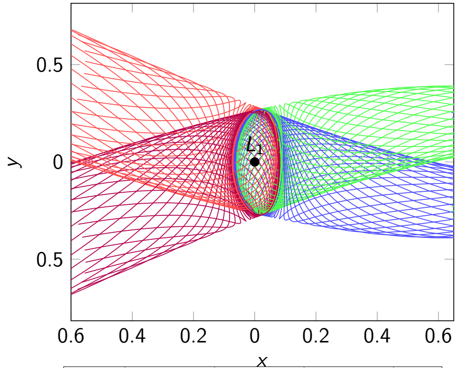
then, we look for $(t_\chi, x_0, \dot{y}_0)$ such that

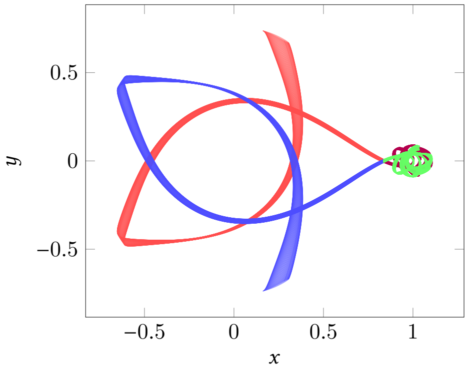
$$\mathcal{P}_\lambda^{\mathcal{E}} : \quad \mathcal{S}_{\mathcal{E}}^{\lambda}(t_\chi, x_0, \dot{y}_0) = \begin{pmatrix} \phi_2(t_\chi/2, \chi_0) \\ \phi_3(t_\chi/2, \chi_0) \\ \mathcal{E}(\chi_0) - \mathcal{E}_\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

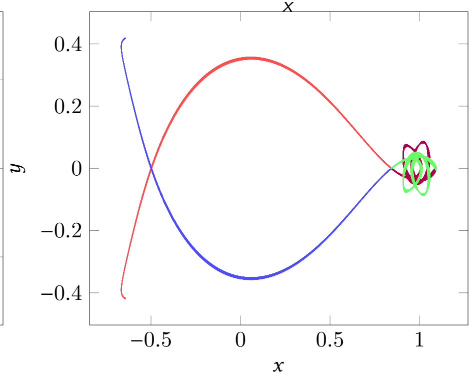
Invariant Manifolds

- ▶ Definition: A *stable* manifold (resp. *instable*) of a periodic orbit is the set of the phase space consisting of all points whose future (resp. past) semi-orbits *converge* to it (asymptotic orbits).

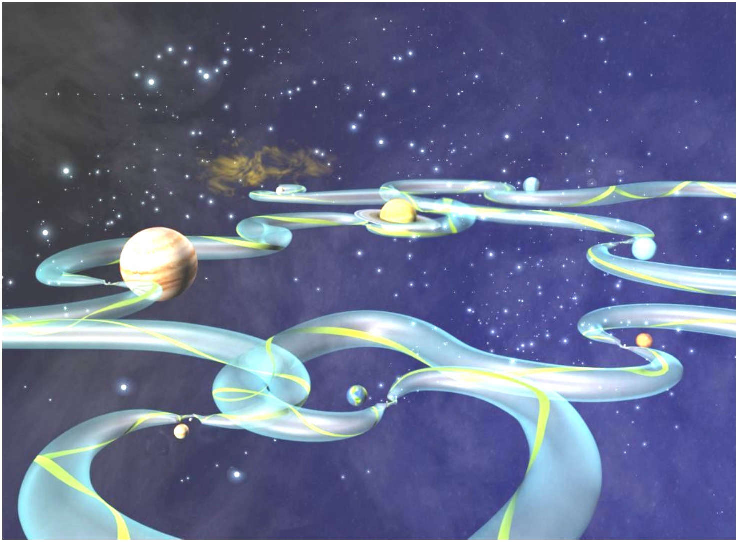
(Koon et *al.*, 2006)





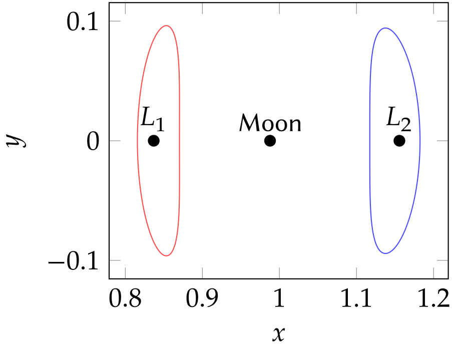


Invariant Manifolds



The Mission

- From one Lyapunov orbit around L_1 to another Lyapunov orbit around L_2 with low thrust:



- The controlled dynamics:

$$\begin{cases} \dot{x} = F_0(x) + \dfrac{\epsilon}{m}\sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \left| u \right|, \end{cases} \qquad F_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $u \in \mathbb{R}^2$, $|u| \leqslant 1$, ϵ is the maximal thrust, β_* is a constant modeling the engines.

- Controllability: see (Caillau, Daoud 2012).
- Similar mission: (Epenoy 2012).

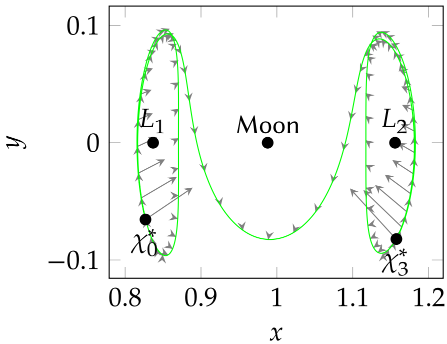
Optimal Control Problem

Goal Problem

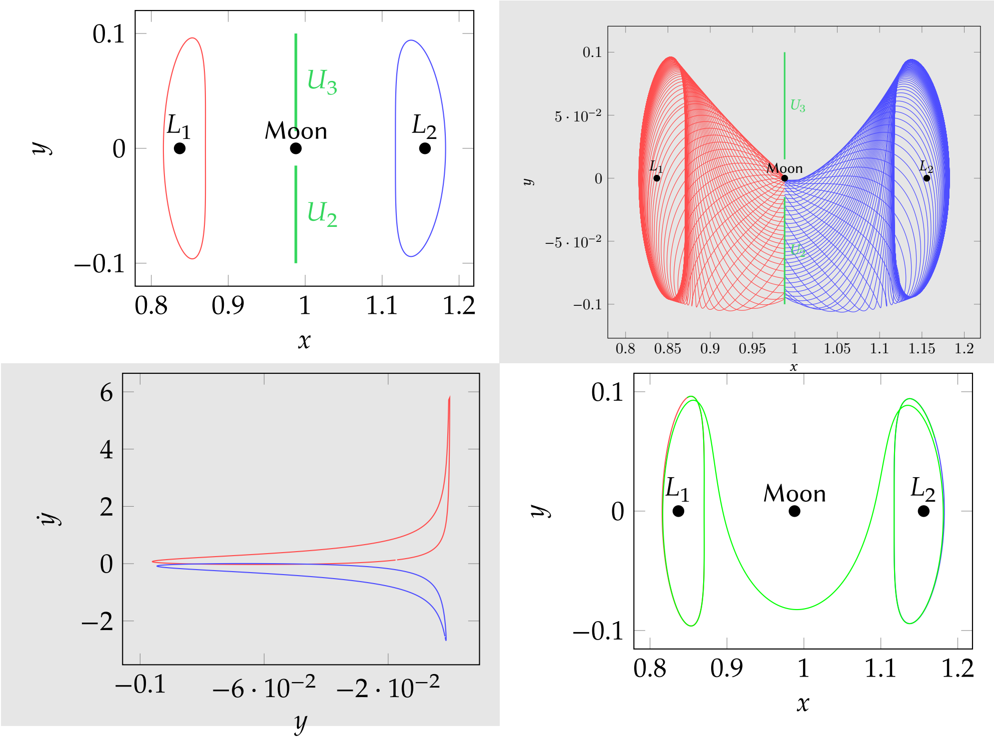
$$\mathcal{P}_g \begin{cases} C_g = \min \int_0^{t_f} \|u\|^2 \, \mathrm{d}t, \\ \dot{x} = F_0(x) + \frac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \, |u| \, , \\ |u| \leq 1, \\ x(0) \in \mathsf{Lya}_1, \text{ and } x(t_f) \in \mathsf{Lya}_2, \\ m(0) = m_0. \end{cases}$$

Tools:

- Use Pontryagin's Maximum Principle (PMP).
- Shooting Method.
- Continuation Method.

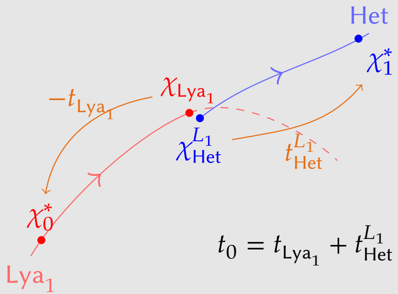


Heteroclinic Orbit



Transfer Lya_1 to Het

Build of a Simpler Problem



$$
\mathcal{P}_{L_1} \begin{cases} \min \int_0^{t_0} \|u\|^2 \, \mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \, |u| \, , \\ |u| \leq 1, \\ x(0) = \chi_0^*, \; m(0) = m_0^*, \\ x(t_0) = \chi_1^*. \end{cases}
$$

Application PMP: Necessary Condition

- $\mathcal{H}(x, m, p, p_m, u)$, normal case, state $(x, m) \in \mathbb{R}^7$, costate $(p, p_m) \in \mathbb{R}^7$

$$
(\dot{x}, \dot{m}) = \frac{\partial H}{\partial(p, p_m)}, \; (\dot{p}, \dot{p}_m) = -\frac{\partial H}{\partial(x, m)}
$$

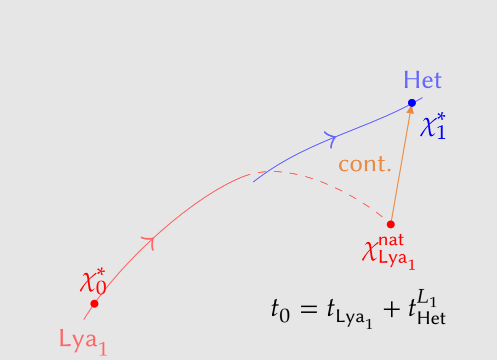
- Maximization condition gives : $u(x, m, p, p_m)$. $(p(0), p_m(0))$?

Shooting function:

$$
\mathcal{S}_{L_1}(p(0), p_m(0)) =
$$

$$
\begin{pmatrix} \phi_{1,\ldots,4}^{\text{ext}}(t_0, \chi_0^*, m_0^*, p(0), p_m(0)) - \chi_1^* \\ \phi_{10}^{\text{ext}}(t_0, \chi_0^*, m_0^*, p(0), p_m(0)) \end{pmatrix}
$$

Transfer Lya_1 to Het



$$\mathcal{P}_{L_1}^{\lambda} \begin{cases} \min \int_0^{t_0} \|u\|^2 \,\mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \,|u|\,, \\ |u| \leq 1, \\ x(0) = \chi_0^*,\ m(0) = m_0^*, \\ x(t_0) = \chi_1^{\lambda} = (1-\lambda)\chi_{\text{Lya}_1}^{\text{nat}} + \lambda \chi_1^*. \end{cases}$$

Continuation:

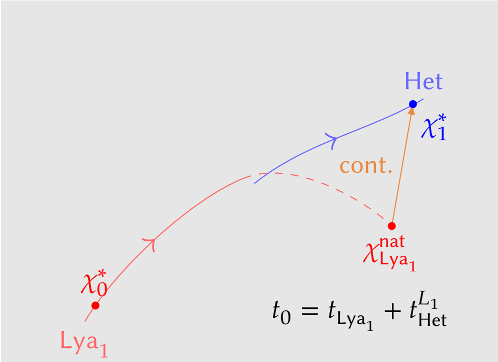
- $(p(0), p_m(0)) = 0$ corresponds to the uncontrolled trajectory.
- Use solution of $\mathcal{P}_{L_1}^{\lambda_{i-1}}$ to initialize $\mathcal{P}_{L_1}^{\lambda_i}$.

Shooting function:

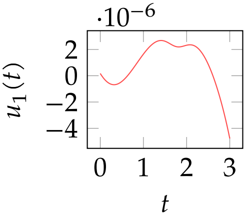
$$\mathcal{S}_{L_1}^{\lambda}(p(0), p_m(0)) =$$

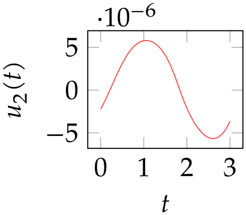
$$\begin{pmatrix} \phi_{1,\dots,4}^{\text{ext}}(t_0, \chi_0^*, m_0^*, p(0), p_m(0)) - \chi_1^{\lambda} \\ \phi_{10}^{\text{ext}}(t_0, \chi_0^*, m_0^*, p(0), p_m(0)) \end{pmatrix}$$

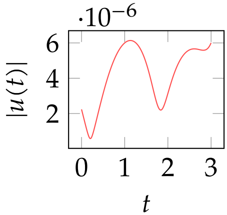
Transfer Lya_1 to Het



$$
\mathcal{P}_{L_1}^{\lambda}
\begin{cases}
\min \int_0^{t_0} \|u\|^2 \, \mathrm{d}t, \\
\dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\
\dot{m} = -\beta_* \epsilon \, |u| \,, \\
|u| \leq 1, \\
x(0) = \chi_0^*, \; m(0) = m_0^*, \\
x(t_0) = \chi_1^{\lambda} = (1-\lambda)\chi_{\mathrm{Lya}_1}^{\mathrm{nat}} + \lambda \chi_1^*.
\end{cases}
$$

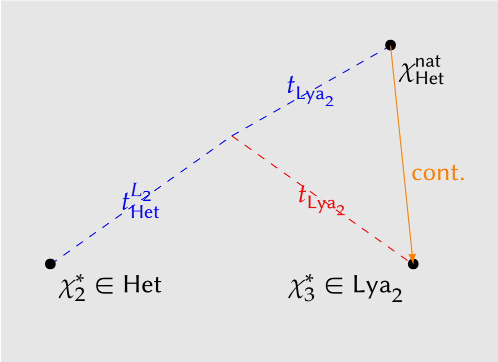




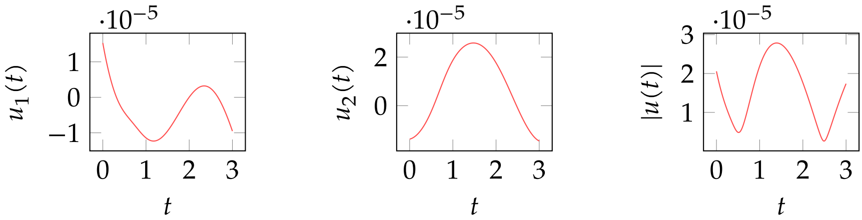


Transfer	Iterations	Cost	$\epsilon \leftrightarrow T_{max}$	Comp. Time
L_1	21	6.30967×10^{-11}	60 N	2.821 s

Transfer Het to Lya_2

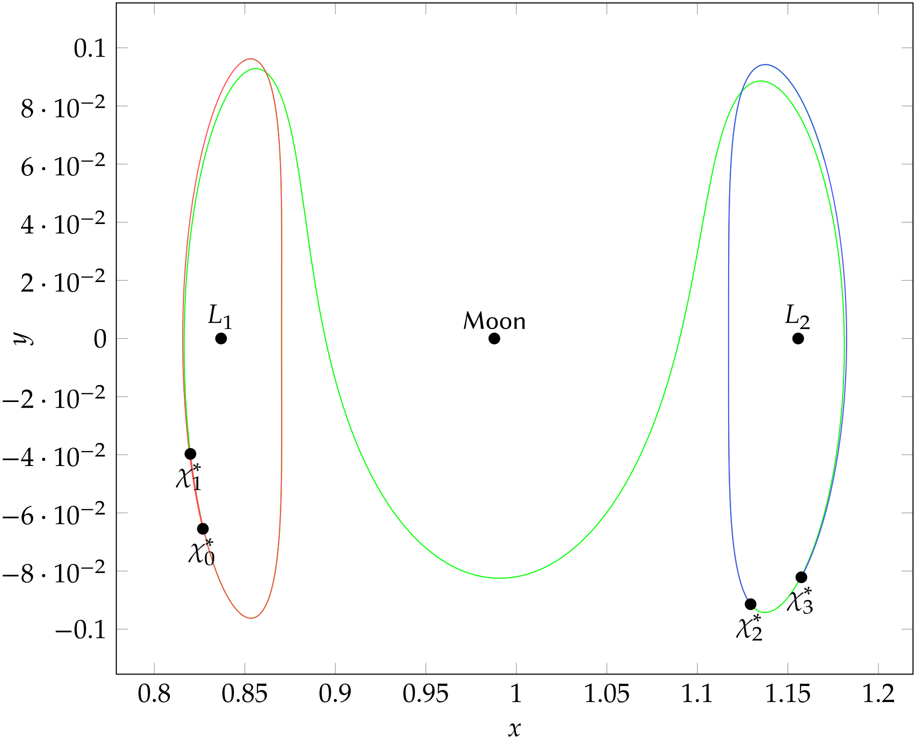


$$\mathcal{P}^{\lambda}_{L_2}\begin{cases} \min \int_0^{t_2} \|u\|^2\, \mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m}\sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon\, |u|\,, \\ |u| \le 1, \\ x(0) = \chi^*_2,\ m(0) = m^*_2, \\ x(t_0) = \chi^{\lambda}_3 = (1-\lambda)\chi^{\mathrm{nat}}_{\mathrm{Het}} + \lambda \chi^*_3. \end{cases}$$



Transfer	Iterations	Cost	$\epsilon \leftrightarrow T_{max}$	Comp. Time
L_2	19	$9.061\,24 \times 10^{-10}$	60 N	1.439 s

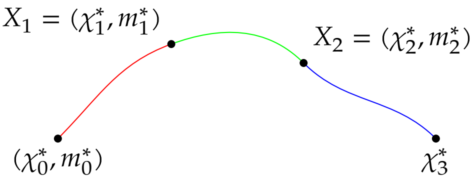
Trajectory with Three Parts



Multiple Shooting

- $t_{\text{tot}} = t_0 + t_1 + t_2$

$$\mathcal{P}_{\text{tot}} \begin{cases} \mathcal{C}_{\text{tot}} = \min \int_0^{t_{\text{tot}}} \|u\|^2 \, \mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \, |u| \,, \\ |u| \leq 1, \\ x(0) = \chi_0^* \in \mathsf{Lya}_1, \; m(0) = m_0^*, \\ x(t_{\text{tot}}) = \chi_3^* \in \mathsf{Lya}_2. \end{cases}$$



- Associate with the corresponding costate initialized with the previous short transfers.

- Shooting function:

$$\mathcal{S}_{\text{tot}}(p(0), p_m(0)) =$$

$$\begin{pmatrix} \phi^{\text{ext}}_{1,\ldots,4}(t_{\text{tot}}, \chi_0^*, m_0^*, p(0), p_m(0)) - \chi_3^* \\ \phi^{\text{ext}}_{10}(t_{\text{tot}}, \chi_0^*, m_0^*, p(0), p_m(0)) \end{pmatrix}$$

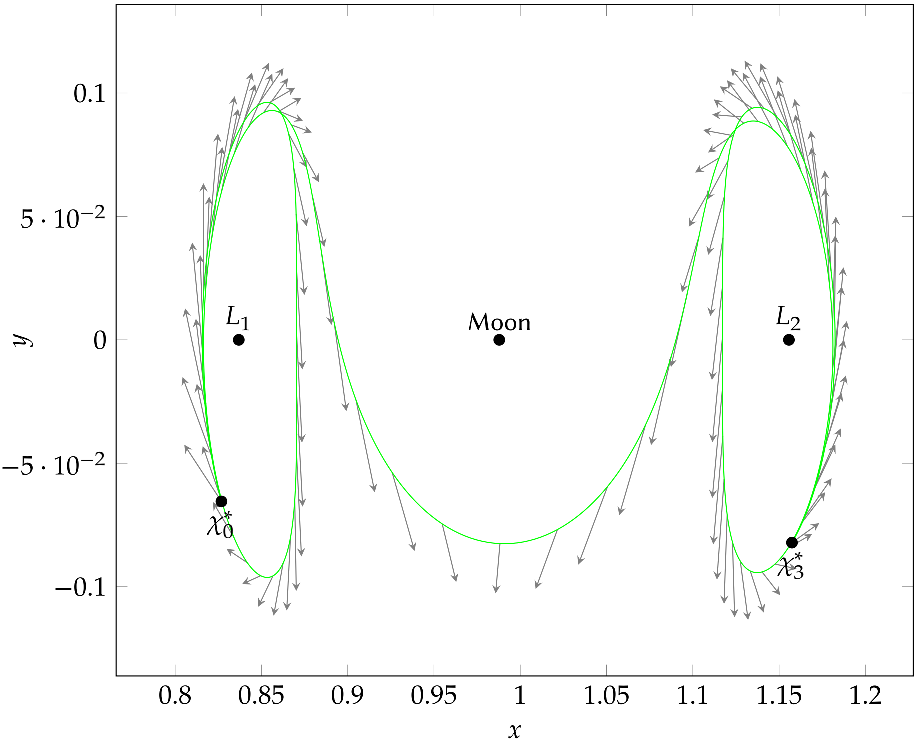
- Initialization? $\Rightarrow$ add nodes

- Use the points (χ_1^*, χ_2^*) on the Heteroclinic orbit

- Shooting function:

$$\mathcal{S}_{\text{multi}}(Z) = \begin{pmatrix} \phi^{\text{ext}}_{1,\ldots,5}(t_0, \chi_0^*, m_0^*, P_0) - X_1 \\ \phi^{\text{ext}}_{6,\ldots,10}(t_0, \chi_0^*, m_0^*, P_0) - P_1 \\ \phi^{\text{ext}}_{1,\ldots,5}(t_1, X_1, P_1) - X_2 \\ \phi^{\text{ext}}_{6,\ldots,10}(t_1, X_1, P_1) - P_2 \\ \phi^{\text{ext}}_{1,\ldots,4}(t_2, X_2, P_2) - \chi_3^* \\ \phi^{\text{ext}}_{10}(t_2, X_2, P_2) \end{pmatrix}.$$

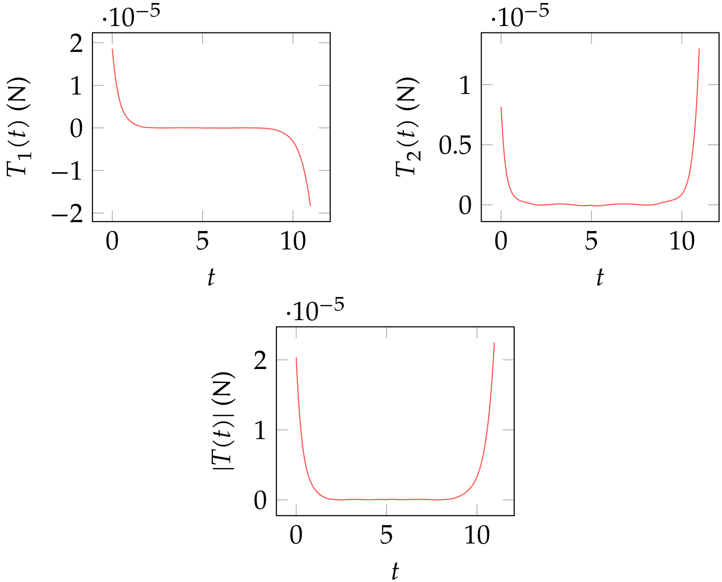
Trajectory


0.1
$5 \cdot 10^{-2}$
0
$-5 \cdot 10^{-2}$
-0.1
y
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
x
L_1
Moon
L_2
χ_0^*
χ_3^*

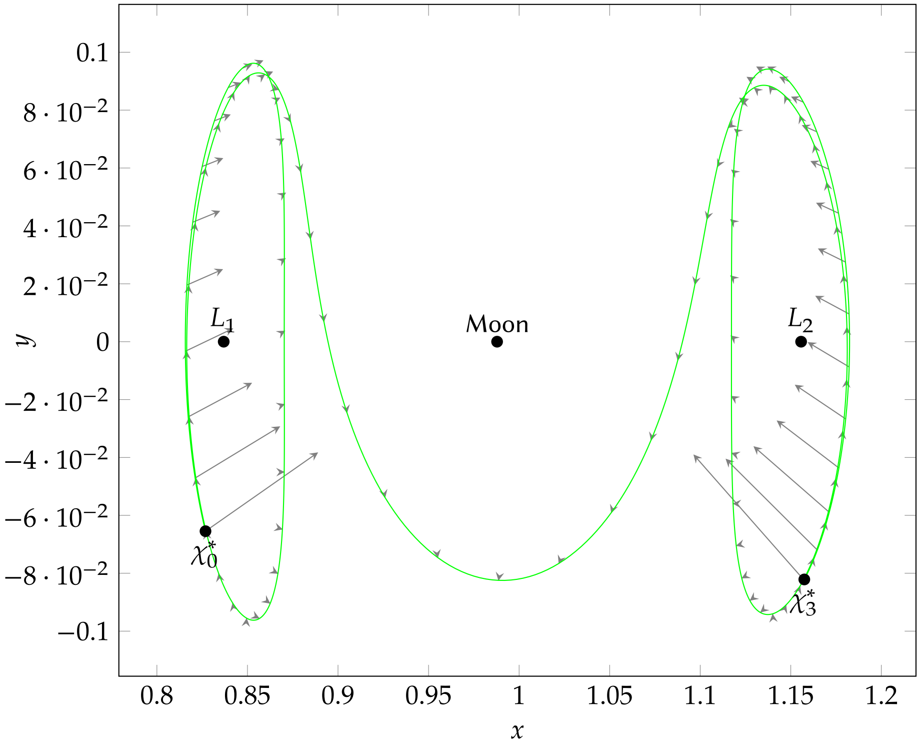
Optimization of Terminal Points

- Pontryagin Maximum Principle applied to the general departure and arrival conditions $x(0) \in \text{Lya}_1$ and $x(t_f) \in \text{Lya}_2$ gives us two transversality conditions:

$$\langle p(0), F_0(x(0)) \rangle = 0 \quad \text{and} \quad \left\langle p(t_f), F_0(x(t_f)) \right\rangle = 0.$$



Trajectory


0.1
$8 \cdot 10^{-2}$
$6 \cdot 10^{-2}$
$4 \cdot 10^{-2}$
$2 \cdot 10^{-2}$
0
$-2 \cdot 10^{-2}$
$-4 \cdot 10^{-2}$
$-6 \cdot 10^{-2}$
$-8 \cdot 10^{-2}$
−0.1
y
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
x
L_1
Moon
L_2
χ_0^*
χ_3^*

Numerical Results

$$
\mathcal{P}_{\text{tot}} \begin{cases} C_{\text{tot}} = \min \int_0^{t_{\text{tot}}} \|u\|^2 \,\mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \,|u|\,, \\ |u| \leq 1, \\ x(0) = \chi_0^* \in \mathsf{Lya}_1,\ m(0) = m_0^*, \\ x(t_{\text{tot}}) = \chi_3^* \in \mathsf{Lya}_2. \end{cases}
$$

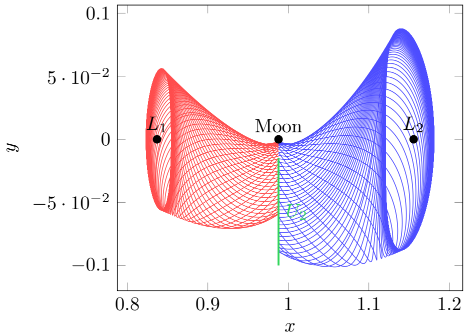
$$
\mathcal{P}_g \begin{cases} C_g = \min \int_0^{t_f} \|u\|^2 \,\mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \,|u|\,, \\ |u| \leq 1, \\ x(0) \in \mathsf{Lya}_1, \text{ and } x(t_f) \in \mathsf{Lya}_2, \\ m(0) = m_0. \end{cases}
$$

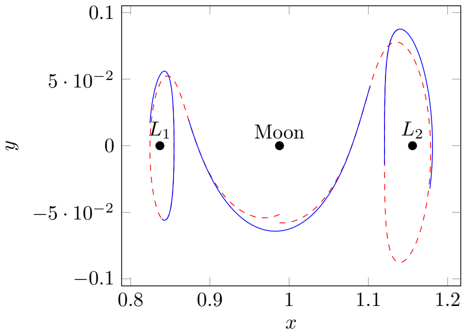
Initial Mass	Transfer time	T_{max}
1500 kg	10.961 39 or 47.67 days	0.3 N

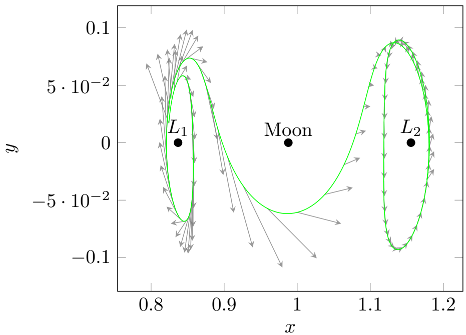
	C_*	Mass of fuel	Exec. Time
$\mathcal{P}_{\text{tot}}$	$1.065\,018\,7 \times 10^{-6}$	0.018 687 8 kg	26.912s
$\mathcal{P}_g$	$2.230\,596\,7 \times 10^{-9}$	$3.670\,958\,9 \times 10^{-4}$ kg	1min 18.64s

Halo to Halo (3D)

- 3D periodic orbit: Halo orbit
- Different energies in L_1 and L_2
- No heteroclinic orbit
- Initialization with 4 parts
- Fast method (~4 min)







Conclusion/Perspectives

Conclusion

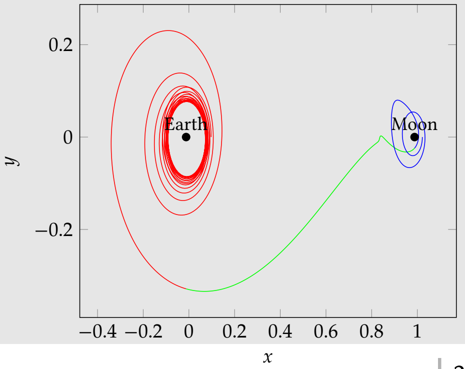
- Very fast method.
- Does not assume any particular structure for the control.
- Very small cost.

Perspectives

- Maximization of final mass: $\int_0^{t_f} |u|$.
- Sun-Earth system, very long time transfer.
- GEO to Moon orbit with *hybrid* optimization coupling direct and indirect methods.

Preprint

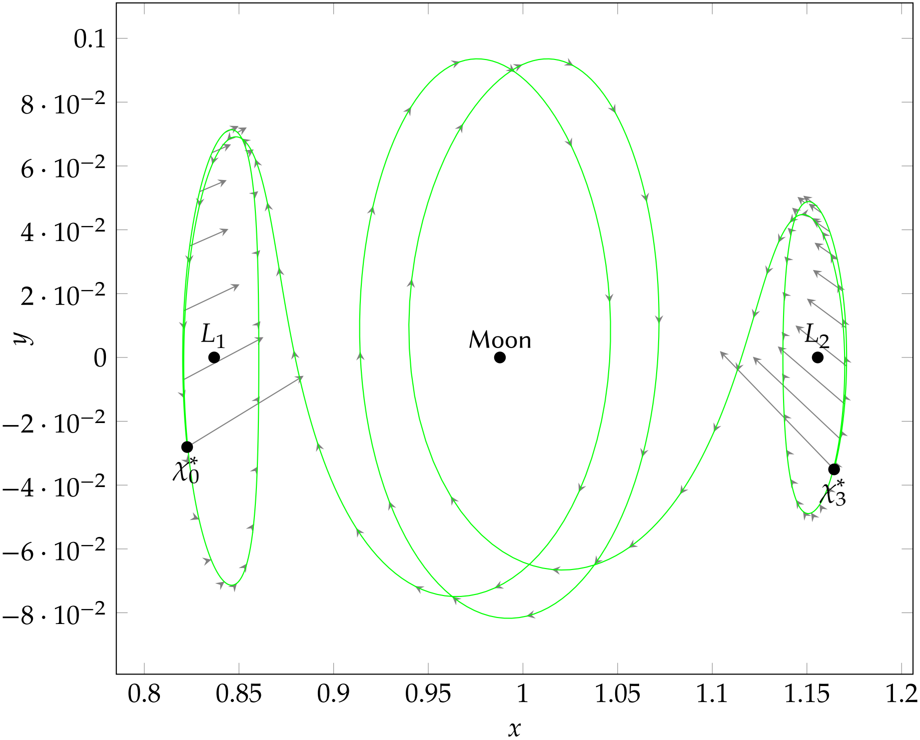
- **M. C., T. Haberkorn, E. Trélat**. *Earth-Moon Lyapunov to Lyapunov Mission: Long Time Duration, Low-Thrust Transfer.* 2015. <hal-01223738>



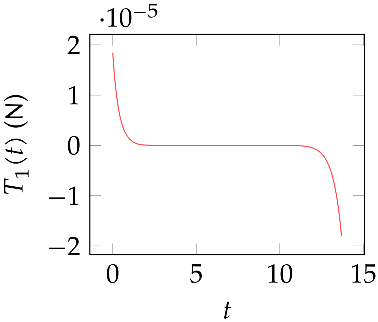
Thank

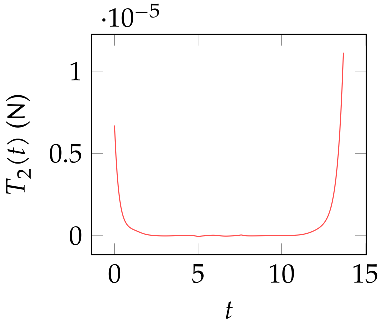
You!

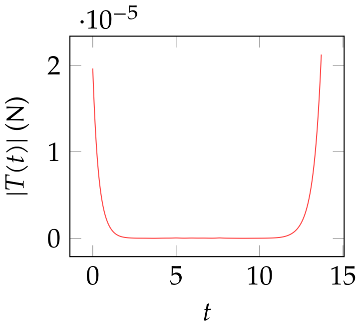
Second Mission – Trajectory



Second Mission – Control







Second Mission – Numerical Results

$$\mathcal{P}_{\text{tot}} \begin{cases} C_{\text{tot}} = \min \int_0^{t_{\text{tot}}} \|u\|^2 \, \mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \, |u| \,, \\ |u| \leq 1, \\ x(0) = \chi_0^* \in \mathsf{Lya}_1, \ m(0) = m_0^*, \\ x(t_{\text{tot}}) = \chi_3^* \in \mathsf{Lya}_2. \end{cases}$$

$$\mathcal{P}_g \begin{cases} C_g = \min \int_0^{t_f} \|u\|^2 \, \mathrm{d}t, \\ \dot{x} = F_0(x) + \dfrac{\epsilon}{m} \sum_{i=1}^{2} u_i F_i(x), \\ \dot{m} = -\beta_* \epsilon \, |u| \,, \\ |u| \leq 1, \\ x(0) \in \mathsf{Lya}_1, \text{ and } x(t_f) \in \mathsf{Lya}_2, \\ m(0) = m_0. \end{cases}$$

Initial Mass	Transfer time	$T_{\max}$
1500 kg	13.6996 or 59.582 days	0.3 N

	C_*	Mass of fuel	Exec. Time
$\mathcal{P}_{\text{tot}}$	$2.463\,890\,5 \times 10^{-8}$	0.003 013 1 kg	44.949s
$\mathcal{P}_g$	$1.969\,593\,4 \times 10^{-9}$	$3.359\,975\,0 \times 10^{-4}$ kg	2min54.79s