Méthodes asynchrones d'éclatement d'opérateurs avec balayage par blocs

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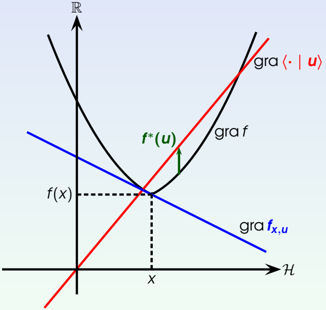
Basic notation

- $\mathcal{H}$: real Hilbert space
- $\Gamma_0(\mathcal{H})$: proper lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$
- $f^*\colon u \mapsto \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - f(x)$ is the Legendre conjugate of f
- The subdifferential of f at $x \in \mathcal{H}$ is

$$\partial f(x) = \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \underbrace{\langle y - x \mid u\rangle + f(x)}_{f_{x,u}(y)} \leqslant f(y)\big\}$$



Fermat's rule:
x minimizes $f \Leftrightarrow 0 \in \partial f(x)$

Monotone operators

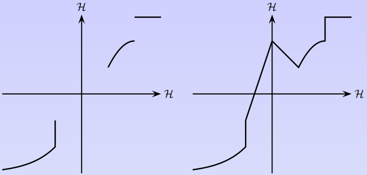
- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone: for every $(x, x^*) \in \mathcal{H}^2$,

$$(x, x^*) \in \operatorname{gra} A \quad \Leftrightarrow \quad (\forall (y, y^*) \in \operatorname{gra} A) \quad \langle x - y \mid x^* - y^* \rangle \geqslant 0$$

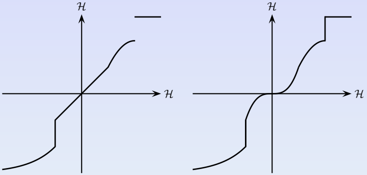
- The resolvent of A, $J_A = (\mathrm{Id} + A)^{-1}\colon \mathcal{H} \to \mathcal{H}$, is firmly nonexpansive and $\operatorname{Fix} J_A = \operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$
- Minty's parametrization:

$$(\forall x \in \mathcal{H}) \quad (J_A x, x - J_A x) = (J_A x, J_{A^{-1}} x) \in \operatorname{gra} A$$

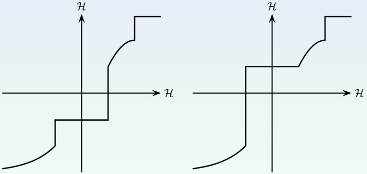
- The problem $0 \in Ax$ covers convex optimization, variational inequalities, complementarity problems, saddle point problems, feasibility problems, fixed point problems, certain Nash equilibria, PDEs, etc
- Example (Moreau's theorem): $f \in \Gamma_0(\mathcal{H})$, $A = \partial f$. Then $\operatorname{zer} A = \operatorname{Argmin} f$ and $J_A = \operatorname{prox}_f$



monotone, not monotone



monotone, max. monotone



max. monotone, max. monotone

Moreau's proximity operator

- In 1962, Jean Jacques Moreau (1923–2014) introduced the ***proximity operator*** of a function $f \in \Gamma_0(\mathcal{H})$

$$J_{\partial f} = \operatorname{prox}_f \colon x \mapsto \underset{y \in \mathcal{H}}{\operatorname{argmin}}\, f(y) + \frac{1}{2}\|x - y\|^2$$

to study problems in unilateral mechanics

- Proximity operators turn out to be very effective tools for modeling and solving data-driven problems:
  - PLC, Convexité et signal, *Proc. SMAI Annual Conf.*, Pompadour, France, April 2001
  - PLC and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, 2005
  - PLC and J.-C. Pesquet, Proximal splitting methods in signal processing, in *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, Springer, New York, 2011

Proximity operators

- Many common convex functions in data processing (statistics, machine learning, image recovery, data denoising, support vector machine, signal processing) have explicit proximity operators:
  - ℓ_1 norm
  - Shatten norm
  - nuclear norm
  - Huber's function
  - Berhu function
  - elastic net regularizer
  - hinge loss
  - Fisher information
  - distance function
  - Vapnik's ε-insensitive loss
  - Burg's entropy
  - ϕ-divergences
  - etc.

Solving monotone inclusions

- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone,
- Problem: solve: $0 \in Ax$
- Conceptual solution methods when A is simple:
  - The proximal point algorithm (implicit):

$$x_{n+1} = (\mathrm{Id} + \gamma_n A)^{-1} x_n = J_{\gamma_n A} x_n, \quad \text{where} \quad \gamma_n > 0.$$

  - If $A\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive ($A^{-1} - \beta\mathrm{Id}$ is monotone), the "steepest descent" iteration (explicit):

$$x_{n+1} = x_n - \gamma_n A x_n, \quad \text{where} \quad 0 < \gamma_n < 2\beta.$$

- For "real" problems **splitting** is required.

Splitting structured problems: 3 basic methods

$A, B \colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, solve $0 \in Ax + Bx$.

- Douglas-Rachford splitting (1979)

$$\left\lfloor \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + z_n - y_n \end{array} \right.$$

- $B \colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive: forward-backward splitting (1979)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 2\beta \\ y_n = x_n - \gamma_n B x_n \\ x_{n+1} = J_{\gamma_n A} y_n \end{array} \right.$$

- $B \colon \mathcal{H} \to \mathcal{H}$ is μ-Lipschitzian: forward-backward-forward splitting (2000)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 1/\mu \\ y_n = x_n - \gamma_n B x_n \\ z_n = J_{\gamma_n A} y_n \\ r_n = z_n - \gamma_n B z_n \\ x_{n+1} = x_n - y_n + r_n \end{array} \right.$$

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in Ax + Bx$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in Ax + L^*B(Lx - r)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ is maximally monotone, $r \in \mathcal{G}$, $L \in \mathscr{L}(\mathcal{H}, \mathcal{G})$

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in Ax + \sum_{k=1}^{K} L_k^* B_k(L_k x - r_k)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{L}(\mathcal{H}, \mathcal{G}_k)$

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in Ax + \sum_{k=1}^{K} L_k^* (B_k \,\square\, D_k)(L_k - r_k x)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{L}(\mathcal{H}, \mathcal{G}_k)$
- $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in Ax + \sum_{k=1}^{K} L_k^*(B_k \,\square\, D_k)(L_k - r_k x) + Cx$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{L}(\mathcal{H}, \mathcal{G}_k)$
- $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$
- $C\colon \mathcal{H} \to \mathcal{H}$ is monotone and μ-Lipschtizian

State-of-the-art in splitting algorithms (PLC, 2013)

find $\overline{x}_1 \in \mathcal{H}_1, \ldots, \overline{x}_m \in \mathcal{H}_m$ such that

$$\begin{cases} z_1^* & \in A_1\overline{x}_1 + \sum_{k=1}^{K} L_{k1}^*\Big((B_k \,\square\, D_k)\Big(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Big)\Big) + C_1\overline{x}_1 \\ & \vdots \\ z_m^* & \in A_m\overline{x}_m + \sum_{k=1}^{K} L_{km}^*\Big((B_k \,\square\, D_k)\Big(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Big)\Big) + C_m\overline{x}_m \end{cases}$$

where:

- $z_i^* \in \mathcal{H}_i$, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ is maximally monotone
- $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{L}(\mathcal{H}, \mathcal{G}_k)$
- $D_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$
- $C_i \colon \mathcal{H}_i \to \mathcal{H}_i$ is monotone and μ_i-Lipschtizian

State-of-the-art in splitting algorithms (PLC, 2013)

- $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_p$
- $\boldsymbol{M}\colon \mathcal{K} \to 2^{\mathcal{K}}\colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto (-z_1^*+A_1x_1) \times \cdots \times (-z_m^*+A_mx_m) \times (r_1+B_1^{-1}v_1^*) \times \cdots \times (r_p+B_p^{-1}v_p^*)$
- $\boldsymbol{Q}\colon \mathcal{K} \to \mathcal{K}\colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto \big(C_1x_1 + \sum_{k=1}^K L_{k1}^* v_k^*, \ldots, C_mx_m + \sum_{k=1}^K L_{km}^* v_k^*, -\sum_{i=1}^m L_{1i}x_i + D_1^{-1}v_1^*, \ldots, \sum_{i=1}^m L_{Ki}x_i + D_K^{-1}v_K^*\big)$
- $\boldsymbol{M}$ and $\boldsymbol{Q}$ are maximally monotone, $\boldsymbol{Q}$ is Lipschitzian, the zeros of $\boldsymbol{M}+\boldsymbol{Q}$ are primal-dual solutions
- Solve $\boldsymbol{0} \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{x}$, where $\boldsymbol{x} = (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*)$ via Tseng's forward-backward-forward splitting algorithm

$$
\left\lfloor
\begin{array}{l}
\boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{Q}\boldsymbol{x}_n \\
\boldsymbol{p}_n = (\mathrm{Id} + \boldsymbol{M})^{-1}\,\boldsymbol{y}_n \\
\boldsymbol{q}_n = \boldsymbol{p}_n - \boldsymbol{Q}\boldsymbol{p}_n \\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n
\end{array}
\right.
$$

in $\mathcal{K}$ to get...

State-of-the-art in splitting algorithms (PLC, 2013)

For $n = 0, 1, \ldots$

$$\varepsilon \leqslant \gamma_n \leqslant (1-\varepsilon)\Big/\Big(\max\Big\{\max_{1\leqslant i\leqslant m}\mu_i, \max_{1\leqslant k\leqslant K}\nu_k\Big\} + \sqrt{\textstyle\sum_{k=1}^K \sum_{i=1}^m \|L_{ki}\|^2}\Big)$$

For $i = 1, \ldots, m$

$$s_{1,i,n} = x_{i,n} - \gamma_n\Big(C_i x_{i,n} + \textstyle\sum_{k=1}^K L_{ki}^* v_{k,n}^*\Big)$$

$$p_{1,i,n} = J_{\gamma_n A_i}(s_{1,i,n} + \gamma_n z_i)$$

For $k = 1, \ldots, K$

$$s_{2,k,n} = v_{k,n}^* - \gamma_n\Big(D_k^{-1} v_{k,n}^* - \textstyle\sum_{i=1}^m L_{ki} x_{i,n}\Big)$$

$$p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + J_{\gamma_n^{-1} B_k}(\gamma_n^{-1} s_{2,k,n} - r_k)\big)$$

$$q_{2,k,n} = p_{2,k,n} - \gamma_n\Big(D_k^{-1} p_{2,k,n} - \textstyle\sum_{i=1}^m L_{ki} p_{1,i,n}\Big)$$

$$v_{k,n+1}^* = v_{k,n}^* - s_{2,k,n} + q_{2,k,n}$$

For $i = 1, \ldots, m$

$$q_{1,i,n} = p_{1,i,n} - \gamma_n\Big(C_i p_{1,i,n} + \textstyle\sum_{k=1}^K L_{ki}^* p_{2,k,n}\Big)$$

$$x_{i,n+1} = x_{i,n} - s_{1,i,n} + q_{1,i,n}$$

State-of-the-art in splitting algorithms (PLC, 2013)

- Each sequence $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to some $\overline{x}_i$ and $(\overline{x}_i)_{1\leqslant i\leqslant m}$ is a solution.
  - PLC, Systems of structured monotone inclusions: Duality, algorithms, and applications, *SIAM J. Optim.*, vol. 23, 2013

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  - PLC, Systems of structured monotone inclusions: Duality, algorithms, and applications, *SIAM J. Optim.*, vol. 23, 2013
- Even in the context of convex optimization, this construction, as most advanced recent splitting methods for optimization, involve monotone operators which are not subdifferentials: **monotone operator theory is needed, even in the limited setting of optimization**

Some limitations of the state-of-the-art

We present a new framework that circumvents simultaneously the limitations of current methods, which require:

- inversions of linear operators or knowledge of bounds on norms of all the L_{ki}
- the proximal parameters must be the same for all the monotone operators
- activation of the resolvents of all the monotone operators: impossible in huge-scale problems
- synchronicity: all resolvent operator evaluations must be computed and used during the current iteration

and in general

- converge only weakly

Asynchronous, block-iterative splitting

- For every $i \in I$ (finite), $\mathcal{H}_i$ a Hilbert space, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ maximally monotone, $z_i^* \in \mathcal{H}_i$
- For every $k \in K$ (finite), $\mathcal{G}_k$ a Hilbert space, $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ maximally monotone, $r_k \in \mathcal{G}_k$, $\mathrm{L}_{ki} \colon \mathcal{H}_i \to \mathcal{G}_k$ linear&bounded
- **Initial problem:** find $(\overline{x}_i)_{i\in I} \in \boldsymbol{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}_i$ such that

$$(\forall i \in I) \quad z_i^* \in A_i\overline{x}_i + \sum_{k\in K} L_{ki}^*\Bigg(B_k\Bigg(\sum_{j\in I} L_{kj}\overline{x}_j - r_k\Bigg)\Bigg)$$

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- **Dual problem:** find $(\overline{v}_k^*)_{k\in K} \in \boldsymbol{\mathcal{G}} = \bigoplus_{k\in K} \mathcal{G}_k$ such that

$$(\forall k \in K) \quad -r_k \in -\sum_{i\in I} L_{ki}\Bigg(A_i^{-1}\Bigg(z_i^* - \sum_{l\in K} L_{li}^*\overline{v}_l^*\Bigg)\Bigg) + B_k^{-1}\overline{v}_k^*$$

Asynchronous, block-iterative splitting

- **Solutions set:** the associated Kuhn-Tucker set

$$\boldsymbol{Z} = \Big\{ \big((\overline{x}_i)_{i\in I}, (\overline{v}_k^*)_{k\in K}\big) \;\Big|\; \overline{x}_i \in \mathcal{H}_i \text{ and } z_i^* - \sum_{k\in K} L_{ki}^* \overline{v}_k^* \in A_i \overline{x}_i, \\ \overline{v}_k^* \in \mathcal{G}_k \text{ and } \sum_{i\in I} L_{ki}\overline{x}_i - r_k \in B_k^{-1}\overline{v}_k^* \Big\}$$

- $\boldsymbol{Z}$ is a closed convex set
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{H}}$ is the set $\boldsymbol{F}$ of primal solutions
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{G}}$ is the set $\boldsymbol{F}^*$ of dual solutions

With proper CQ, this framework includes..

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{(x_i)_{i\in I}\in\boldsymbol{\mathcal{H}}}{\text{minimize}} \sum_{i\in I}\big(f_i(x_i) - \langle x_i \mid z_i^*\rangle\big) + \sum_{k\in K} g_k\Big(\sum_{i\in I} L_{ki}x_i - r_k\Big)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki}\colon \mathcal{H}_i \to \mathcal{G}_k$ linear&bounded

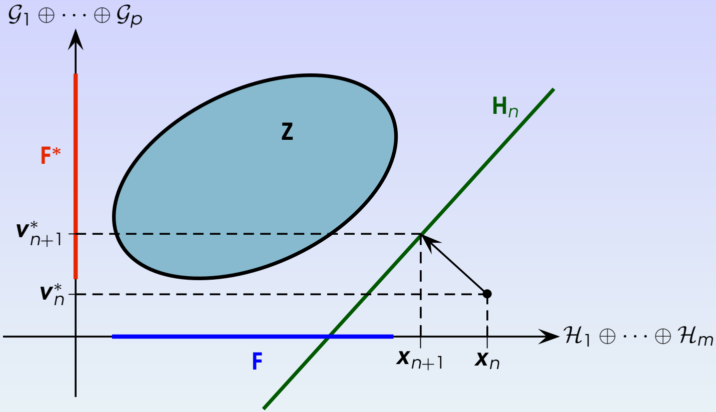
- Let $\boldsymbol{F}^*$ be the set of solutions to the dual problem

$$\underset{(v_k^*)_{k\in K}\in\bigoplus_{k\in K}\mathcal{G}_k}{\text{minimize}} \sum_{i\in I} f_i^*\Big(z_i^* - \sum_{k\in K} L_{ki}^* v_k^*\Big) + \sum_{k\in K}\big(g_k^*(v_k^*) + \langle v_k^* \mid r_k\rangle\big)$$

- Associated Kuhn-Tucker set

$$\boldsymbol{Z} = \Big\{\big((\overline{x}_i)_{i\in I}, (\overline{v}_k^*)_{k\in K}\big) \;\Big|\; \overline{x}_i \in \mathcal{H}_i \text{ and } z_i^* - \sum_{k\in K} L_{ki}^*\overline{v}_k^* \in \partial f_i(\overline{x}_i),$$
$$\overline{v}_k^* \in \mathcal{G}_k \text{ and } \sum_{i\in I} L_{ki}\overline{x}_i - r_k \in \partial g_k^*(\overline{v}_k^*)\Big\}$$

Asynchronous block-iterative proximal splitting I



- Choose suitable points in the graphs of $(A_i)_{i\in I}$ and $(B_k)_{k\in K}$ to construct a half-space $\mathbf{H}_n$ containing $\mathbf{Z}$
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}^*_{n+1}) = P_{\mathbf{H}_n}(\boldsymbol{x}_n, \boldsymbol{v}^*_n) \rightharpoonup (\boldsymbol{x}, \boldsymbol{v}^*) \in \mathbf{Z} \subset \mathbf{F} \times \mathbf{F}^*$

Main novelties

- **Block iterations:** At iteration n, we require calculation of new points in the graphs of only some the operators $(A_i)_{i\in I_n\subset I}$ and $(B_k)_{k\in K_n\subset K}$. The control sequences $(I_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ dictate how frequently the various operators are used.
- **Asynchronicity:** A new point $(a_{i,n}, a^*_{i,n}) \in \operatorname{gra} A_i$ being incorporated into the calculations at iteration n may be based on data $x_{i,c_i(n)}$ and $(v^*_{k,c_i(n)})_{k\in K}$ available at some possibly earlier iteration $c_i(n) \leqslant n$. Therefore, the calculation of $(a_{i,n}, a^*_{i,n})$ could have been initiated at iteration $c_i(n)$, with its results becoming available only at iteration n. Likewise, for $(b_{k,n}, b^*_{k,n}) \in \operatorname{gra} B_k$.

Also:

- No knowledge of the $\|L_{ki}\|$s is required
- No linear operator inversion is required
- No bounds required on the proximal parameters

Asynchronous block-iterative proximal splitting I

for $n = 0, 1, \ldots$

- for every $i \in I_n$
  - $l_{i,n}^* = \sum_{k \in K} L_{ki}^* v_{k,c_i(n)}^*$
  - $(a_{i,n}, a_{i,n}^*) = \Big(J_{\gamma_{i,c_i(n)} A_i}\big(x_{i,c_i(n)} + \gamma_{i,c_i(n)}(z_i - l_{i,n}^*)\big), \gamma_{i,c_i(n)}^{-1}(x_{i,c_i(n)} - a_{i,n}) - l_{i,n}^*\Big)$
- for every $i \in I \smallsetminus I_n$
  - $(a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*)$
- for every $k \in K_n$
  - $l_{k,n} = \sum_{i \in I} L_{ki} x_{i,d_k(n)}$
  - $(b_{k,n}, b_{k,n}^*) = \Big(r_k + J_{\mu_{k,d_k(n)} B_k}\big(l_{k,n} + \mu_{k,d_k(n)} v_{k,d_k(n)}^* - r_k\big), v_{k,d_k(n)}^* + \mu_{k,d_k(n)}^{-1}(l_{k,n} - b_{k,n})\Big)$
- for every $k \in K \smallsetminus K_n$
  - $(b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*)$
- $\big((t_{i,n}^*)_{i \in I}, (t_{k,n})_{k \in K}\big) = \big((a_{i,n}^* + \sum_{k \in K} L_{ki}^* b_{k,n}^*)_{i \in I}, (b_{k,n} - \sum_{i \in I} L_{ki} a_{i,n})_{k \in K}\big)$
- $\tau_n = \sum_{i \in I} \|t_{i,n}^*\|^2 + \sum_{k \in K} \|t_{k,n}\|^2$
- if $\tau_n > 0$
  - $$\theta_n = \frac{\lambda_n}{\tau_n} \max\Big\{0, \sum_{i \in I}\big(\langle x_{i,n} \mid t_{i,n}^*\rangle - \langle a_{i,n} \mid a_{i,n}^*\rangle\big) + \sum_{k \in K}\big(\langle t_{k,n} \mid v_{k,n}^*\rangle - \langle b_{k,n} \mid b_{k,n}^*\rangle\big)\Big\}$$
- else $\theta_n = 0$
- for every $i \in I$
  - $x_{i,n+1} = x_{i,n} - \theta_n t_{i,n}^*$
- for every $k \in K$
  - $v_{k,n+1}^* = v_{k,n}^* - \theta_n t_{k,n}$

Convergence

- $(I_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of I, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of K such that $I_0 = I$, $K_0 = K$, and

$$(\forall n \in \mathbb{N}) \quad \left(\bigcup_{j=n}^{n+M-1} I_j = I \quad \text{and} \quad \bigcup_{j=n}^{n+M-1} K_j = K \right). \tag{1}$$

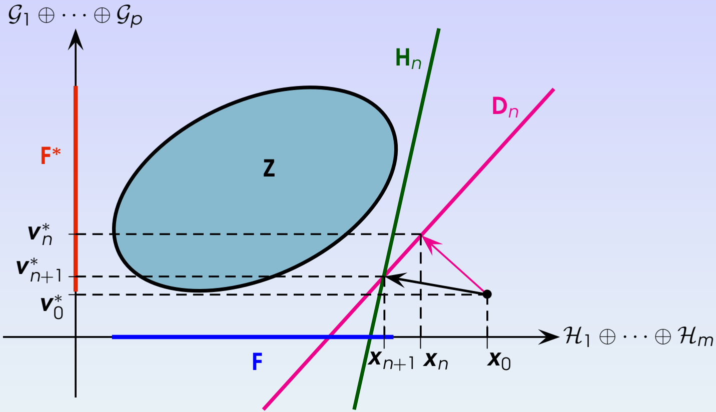
- $(c_i(n))_{n\in\mathbb{N}}$ and $(d_k(n))_{n\in\mathbb{N}}$ are sequences in $\mathbb{N}$ such that

$$(\forall i \in I) \;\; n-D \leqslant c_i(n) \leqslant n \quad \text{and} \quad (\forall k \in K) \;\; n-D \leqslant d_k(n) \leqslant n$$

- $\varepsilon \in \,]0,1[$ and $(\gamma_{i,n})_{n\in\mathbb{N}}$ and $(\mu_{k,n})_{n\in\mathbb{N}}$ are sequences in $[\varepsilon, 1/\varepsilon]$.

Set $x_n = (x_{i,n})_{i\in I}$ and $v_n^* = (v_{k,n}^*)_{k\in K}$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{x} \in \boldsymbol{F}$, $(v_n^*)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{v} \in \boldsymbol{F}^*$, and $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$.

Asynchronous block-iterative proximal splitting II



- Construct $\mathbf{H}_n$ as before
- The half-space $\mathbf{D}_n$ satisfies $(\boldsymbol{x}_n, \boldsymbol{v}_n^*) = P_{\mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*)$
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}_{n+1}^*) = P_{\mathbf{H}_n \cap \mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \to P_{\mathbf{Z}}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \in \mathbf{F} \times \mathbf{F}^*$

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