Projections alternées pour des contraintes de convexité généralisées

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Outline

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An example

The following problem arises in economics [Rochet, Choné '98]:

$$\min_{\substack{u \text{ convex} \\ u \geq 0}} \int \left[c(\nabla u(x)) - x \cdot \nabla u(x) + u(x) \right] \mathrm{d}\mu(x)$$

Convexity constraint

- arises in various contexts (e.g. Newton's least resistance)
- no tractable Euler-Lagrange equation
- challenging for numerical methods:
  [Carlier et al. '01], . . . , [Ekeland, Moreno-Bromberg '10],
  [Oberman '13], [Mérigot, Oudet '14], [Mirebeau '15]

A generalization

Definition (b-convexity)

A function u is b-convex if for some function v and for all x

$$u(x) = \sup_{y \in Y} \{ b(x, y) - v(y) \}$$

This notion is crucial in optimal transport [Gangbo, McCann '96].
For $b(x, y) = x \cdot y$, we recover the notion of convexity.

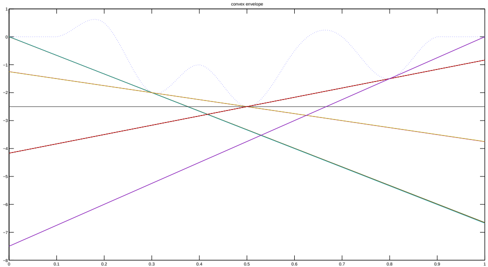

Figure: Convex vs b-convex envelope

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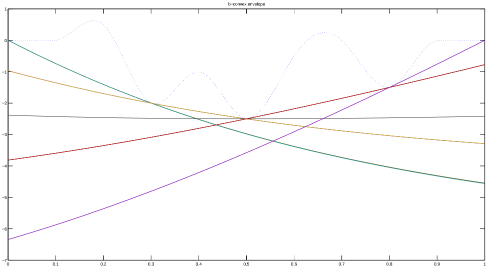

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b-convexity constraints

- arise in economics [Carlier '01]
- the convexity of $\{u \text{ } b\text{-convex}\}$ requires assumptions on b [Figalli, Kim, McCann '11]
- no numerical method so far

The principal-agent problem

A principal (e.g. monopolist) faces

- a population of agents: agents of type $x \in X$ are present according to a distribution $\mathrm{d}\mu(x)$;
- a range of products: products of type $y \in Y$ have value $b(x,y)$ for agents of type x and cost $c(y)$ for the principal.

She is looking for

- a contract menu $(y,p)\colon X \to Y \times \mathbb{R} \cup \{+\infty\}$

which specifies that

- agents x buy the product $y(x)$ at the price $p(x)$

and which maximizes her profit $\displaystyle\int_X [p(x) - c(y(x))]\, \mathrm{d}\mu(x)$.

Incentive compatibility constraint

The utility (to be maximized) of agents of type x is

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- zero for not buying anything (outside option).

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Adverse selection

Agents actually behave according to a contract menu (y, p) if it is incentive compatible and provide non-negative utility:

$$
\begin{aligned}
b(x, y(x)) - p(x) &\geq b(x, y(x')) - p(x') \quad \forall (x, x'), \qquad \text{(IC)} \\
b(x, y(x)) - p(x) &\geq 0 \quad \forall x. \qquad \text{(IR)}
\end{aligned}
$$

The principal's problem is then the following optimization problem:

$$
\min_{(y,p)} \int_X \left[c(y(x)) - p(x)\right] \mathrm{d}\mu(x) \quad \text{subject to (IC)-(IR).}
$$

Change of variables

Given (y, p), we define u by $u(x) := b(x, y(x)) - p(x)$ for all x.
The incentive compatibility constraint (IC) becomes

$$u(x') - u(x) \geq b(x', y(x)) - b(x, y(x)) \quad \forall (x, x') \tag{IC'}$$

If (y, u) satifies (IC'), then

- $u(x) = \sup_y \{b(x, y) - v(y)\}$ for $v(y(x)) := p(x)$,
- u is b-convex, $\nabla u(x) = \frac{\partial b}{\partial x}(x, y(x))$ if u is differentiable at x.

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Assumption 1 The b-exponential map y_b is well defined by

$$y = y_b(x, q) \iff q = \frac{\partial b}{\partial x}(x, y).$$

Given (y, u), we define q by $q(x) := \frac{\partial b}{\partial x}(x, y(x))$ for all x,
we replace $y(x)$ by $y_b(x, q(x))$ in (IC') and we introduce

$$\Gamma_b(x', x, q) := b(x', y_b(x, q)) - b(x, y_b(x, q)).$$

The convexity issue

We therefore consider, more generaly, the following problem:

$$\min_{(u,q)} \int_X L(x, u(x), q(x))\mathrm{d}x \quad \text{subject to}$$
$$u(x') - u(x) \geq \Gamma_b(x', x, q(x)) \quad \forall (x, x') \qquad \text{(IC")}$$

It is convex if $L(x, \cdot, \cdot)$ is convex for all x and if b satisfies

Assumption 2 The function $\Gamma_b(x', x, \cdot)$ is convex for all x, x'.

[Figalli, Kim, McCann '11], [Carlier, D. '15]

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Remarks

- If (u, q) is feasible, then u is b-convex and $q = \nabla u$ a.e.
- For $b(x, y) = x \cdot y$, $\Gamma_b(x', x, q) = (x' - x) \cdot q$ and the convexity constraint (IC'') is as in [Ekeland, Moreno-Bromberg '10].

A tractable class of b

We consider perturbations of the scalar product

$$b(x,y) = x \cdot y + f(x)g(y) \quad \text{on } X \times \mathbb{R}^d$$

with f, g convex and C^1, $g \geq 0$, $\inf_{x,y} \nabla f(x) \cdot \nabla g(y) > -1$;

- $\Gamma_b(x', x, q) = (x' - x) \cdot q + D_f(x', x) g(y_b(x, q))$ where
- $D_f \geq 0$ is the Bregman divergence associated to f,
- $g(y_b(x, q))$ is the solution to the scalar equation

$$\lambda = g(q - \lambda \nabla f(x))$$

and can be shown to be convex w.r.t. q.

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Numerical implementation (to come)

- $f(x) = \sqrt{1 + |x|^2}$, $g(y) = \sqrt{1 + |y|^2}$
- closed form for $g(y_b(x, q))$

Outline

Motivation and assumptions

Projection problems over *b*-convex functions

Dykstra's algorithm and numerical results

Quadratic integrands

We consider a projection problem over b-convex functions, that we discretize for any $\{x_k\}_{1\leq k\leq N} \subset X$ as follows:

$$\min_{((u_k)_k,(q_k)_k)} \sum_{k=1}^{N} \left[\frac{\alpha_k}{2} |q_k - \bar{q}_k|^2 + \frac{\beta_k}{2} |u_k - \bar{u}_k|^2 \right]$$

$$\text{subject to} \quad u_i - u_j \geq \Gamma_b(x_i, x_j, q_j) \quad \forall (i,j) \qquad \text{(IC'')}$$

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Solving these convex optimization problems, we get interior and convergent approximations of the continuous solution by setting

$$u^N(x) := \max_{1\leq k\leq N} \{u_k + \Gamma_b(x, x_k, q_k)\}, \quad q^N(x) := \nabla u^N(x).$$

[Ekeland, Moreno-Bromberg '10] (convex case), [Carlier, D. '15]

Formulation as a projection problem

Introducing the convex sets, for all (i, j), in $\mathbb{R}^N \times \mathbb{R}^{dN}$

$$C_{i,j} := \{(u, q) : \; u_i - u_j \geq \Gamma_b(x_i, x_j, q_j)\},$$

the discrete problem can be seen as a projection problem

- onto their intersection $C := \bigcap_{(i,j)} C_{i,j}$
- for the weighted Euclidean distance $D_{\alpha,\beta}$

where is to be found $P_C(\bar{u}, \bar{q})$, solution in $\mathbb{R}^N \times \mathbb{R}^{dN}$ to

$$\min_{(u,q)\in C} D_{\alpha,\beta}\big((u, q), (\bar{u}, \bar{q})\big).$$

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We solve it iteratively by Dykstra's algorithm [Boyle, Dykstra '86]: at each step we compute an elementary projection $P_{C_{i,j}}$ and then cycle over the N^2 convex sets; the sequence that we get converges.

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Elemantary projections

Any elementary projection $P_{C_{i,j}}(\hat{u}, \hat{q},)$ coincides with $(\hat{q}, \hat{u})$ up to (u_i, u_j, q_j), solution to

$$\min_{(u_i, u_j, q_j)} \frac{\alpha_j}{2} |q_j - \hat{q}_j|^2 + \frac{\beta_i}{2} |u_i - \hat{u}_i|^2 + \frac{\beta_j}{2} |u_j - \hat{u}_j|^2$$

$$\text{subject to} \quad u_i - u_j \geq \Gamma_b(x_i, x_j, q_j)$$

- optimization problem in $\mathbb{R}^2 \times \mathbb{R}^d$ (instead of $\mathbb{R}^N \times \mathbb{R}^{dN}$)
- single scalar inequality constraint (instead of N^2)

We solve this problem by determining a primal-dual solution to

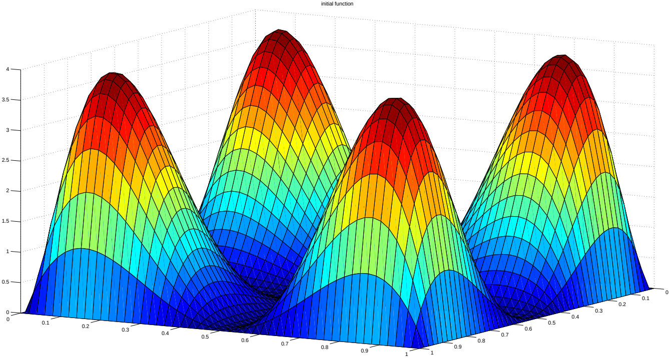
- the necessary and sufficient (KKT) optimality conditions
- by a Newton method when the constraint is active.

The projections are explicit in the convex case $b(x, y) = x \cdot y$.

H^1 projection on b-convex functions

$$\min_{(u,q)} \int_X \left[\frac{1}{2} \left| q(x) - \nabla \bar{u}(x) \right|^2 + \frac{1}{2} \left| u(x) - \bar{u}(x) \right|^2 \right] \mathrm{d}x$$

subject to (IC")



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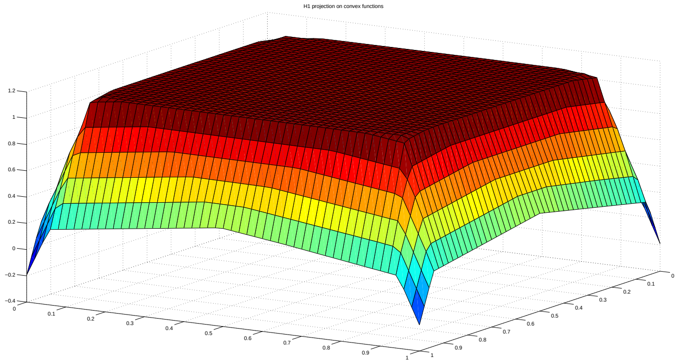

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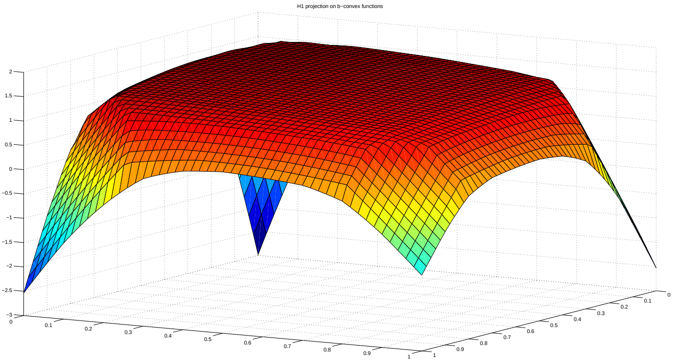

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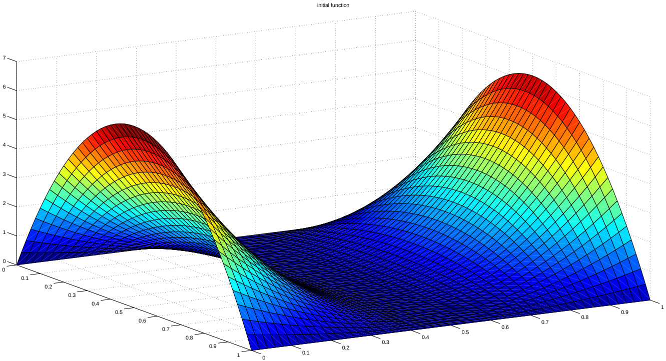

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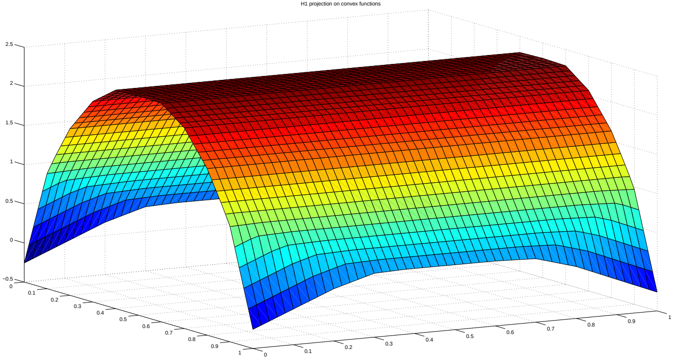

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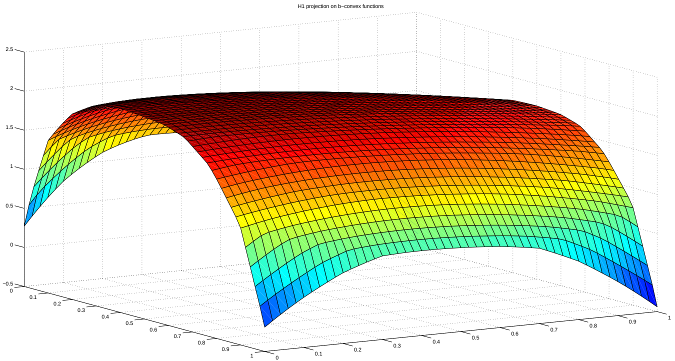
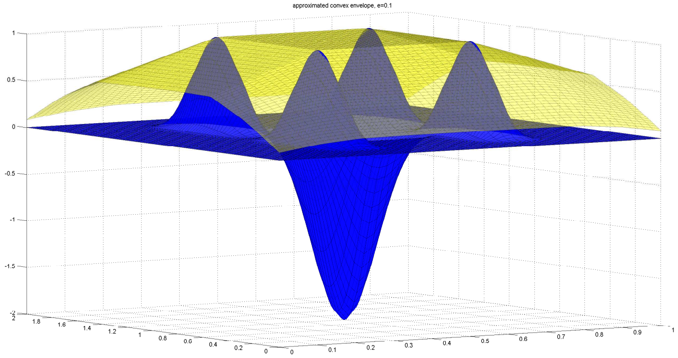


Approximated b-convex envelope

$$\min_{(u,q)} \int_X \left[\frac{\varepsilon}{2} |q(x)|^2 + \frac{1}{2} |u(x) - \bar{u}(x)|^2 \right] \mathrm{d}x$$

subject to (IC"), $u(x) \leq \bar{u}(x)\ \forall x$



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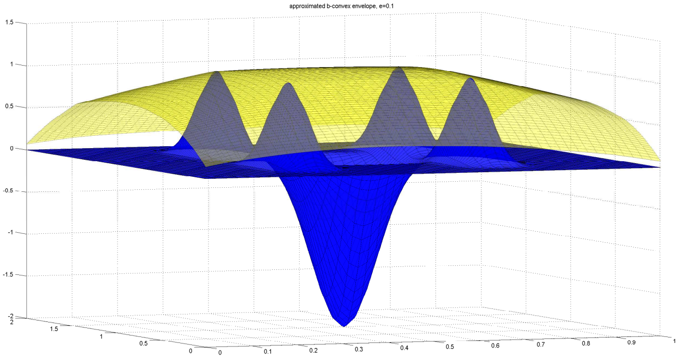

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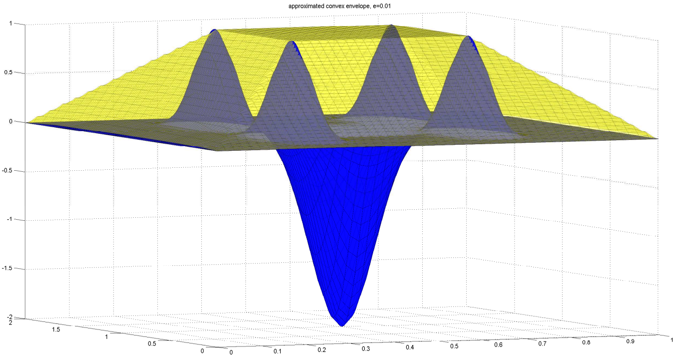

Approximated b-convex envelope

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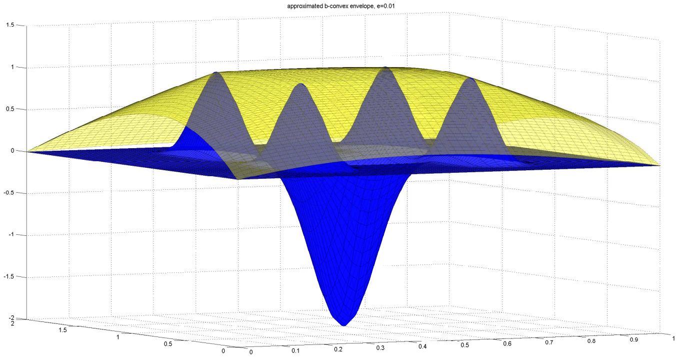
subject to (IC''), $u(x) \leq \bar{u}(x)\ \forall x$



Principal-agent problem

In the convex case $b(x,y) = x \cdot y$ and $c(y) = \frac{1}{2}|y|^2$,

$$\int_X \left[\frac{1}{2}|q(x)|^2 - x \cdot q(x) + u(x)\right] \mathrm{d}\mu(x)$$

can be regularized to fit into the projections framework as

$$\int_X \left[\frac{1}{2}|q(x) - x|^2 + \frac{\varepsilon}{2}|u(x) + \frac{1}{\varepsilon}|^2\right] \mathrm{d}\mu(x).$$

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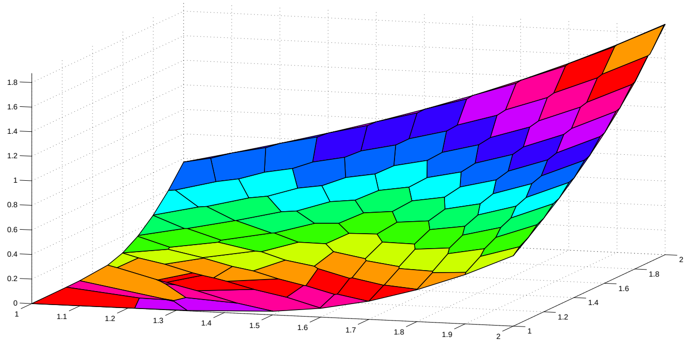
For more general $b(x,y) = x \cdot y + f(x)g(y)$, when the objective

$$\int_X \left[\frac{1}{2}|y_b(x, q(x))|^2 - b(x, y_b(x, q(x)) + \frac{\varepsilon}{2}|u(x) + \frac{1}{\varepsilon}|^2\right] \mathrm{d}\mu(x)$$

is a Bregman distance (in particular strictly convex) in (q, u), an extension of Dykstra's algorithm is available [Baushcke, Lewis '98].

Principal-agent problem

Indirect utility of the agents in the convex case
(μ uniform distribution)



Principal-agent problem

Distribution of the types of products sold in the convex case (μ uniform distribution)