Exact algorithms for linear matrix inequalities

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SMAI-MODE, 2016

Spectrahedra and LMI

$A_0, A_1, \ldots, A_n$ are $m \times m$ real symmetric matrices

Spectrahedron: $\mathscr{S} = \{x \in \mathbb{R}^n : A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0\}$

It is **basic semi-algebraic** since, if

$$\det(A(x) + tI_m) = f_m(x) + f_{m-1}(x)t + \cdots + f_1(x)t^{m-1} + t^m$$

then $\mathscr{S} = \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1, \ldots, m\}$. $A(x) \succeq 0$ is called an **LMI**.

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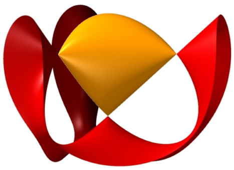

Figure: The Cayley spectrahedron

SDP : linear optimization over $\mathscr{S}$ (*i.e.* over **LMI**)

$$\mathscr{S} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \succeq 0 \right\}$$

Why exact algorithms?

1. It is **Hard** to compute low-rank solutions to SDP

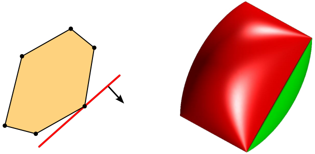

Figure: "**Low-rank**" points : they minimize a cone of linear forms

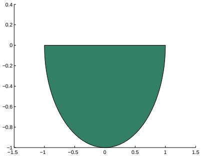

Figure: SEDUMI returns a floating point approximation of $(0, 0)$ when maximizing x_2

2. The interior of $\mathscr{S}$ can be **empty** $\longrightarrow$ **Interior point algorithms could fail**

$$\begin{bmatrix} 0 & x_1 & \frac{1}{2}(1 - x_4) \\ x_1 & x_2 & x_3 \\ \frac{1}{2}(1 - x_4) & x_3 & x_4 \end{bmatrix} \succeq 0$$

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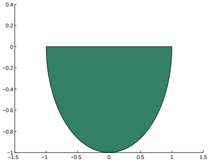

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Main motivations for the design of **exact algorithms**:

1. Can we manage **algebraic constraints** such as rank defects?
2. Can we handle **degenerate** non-full-dimensional examples?
3. **Consequence:**
   The output is a point whose coordinates may be **real algebraic numbers**

$$(q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[t] \quad \rightarrow \quad \left\{ \left(\frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)} \right) : q(t) = 0 \right\}$$

State of the art

Decision / Sampling problem for real algebraic or semi-algebraic sets

Cylindrical Algebraic Decomposition

Tarski (1948), Seidenberg, Cohen, ...
Collins (1975) in $\mathcal{O}((2m)^{2^{2n+8}} m^{2^{n+6}}), \dots$

Critical Points Method **local extrema of algebraic maps f on $\mathscr{S}$**

Grigoriev, Vorobjov (1988) first singly exp: $m^{\mathcal{O}(n^2)}$
Renegar (1992), Heintz Roy Solernó (1989,1993), Basu Pollack Roy (1996,...)
linear exponent $m^{\mathcal{O}(n)}$

Polar varieties **local extrema of linear projections π on $\mathscr{S}$**

Bank, Giusti, Heintz, Mbakop, Pardo (1997,...)
Safey El Din, Schost (2003,2004) regular in $\mathcal{O}(m^{3n})$, singular in $\mathcal{O}(m^{4n})$

The goal was:
Better results for spectrahedra?
How to take advantage of the structure?

Complexity of SDP

Special case of SDP

Khachiyan, Porkolab (1996) decide LMI-feasibility in time

$$\boxed{\mathcal{O}(nm^4) + m^{\mathcal{O}(\min\{n,m^2\})}} \quad \text{on} \quad (\ell m^{\mathcal{O}(\min\{n,m^2\})})\text{-bit numbers}$$

($\ell =$ input bit-size)

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✓ **Positive aspects:**

1. No assumptions, Deterministic
2. Binary complexity

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✗ **Main drawbacks:**

1. It relies on Quantifier Elimination
2. Too large constant in the exponent

Low rank positive semidefinite matrices

Define:

For any $A(x)$ (not nec. symmetric): $\mathcal{D}_r = \{x \in \mathbb{C}^n : \operatorname{rank} A(x) \leq r\}$

For $A(x)$ symmetric, and $\mathscr{S} \neq \emptyset$: $r(A) = \min\{\operatorname{rank} A(x) \mid x \in \mathscr{S}\}$

So one has nested sequences

$$\mathcal{D}_0 \subset \cdots \subset \mathcal{D}_{m-1}$$
$$\mathcal{D}_0 \cap \mathbb{R}^n \subset \cdots \subset \mathcal{D}_{m-1} \cap \mathbb{R}^n$$

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Smallest Rank Property — Henrion-N.-Safey El Din 2015

$A(x)$ symmetric, and $\mathscr{S} \neq \emptyset$. Let $\mathcal{C}$ be a conn. comp. of $\mathcal{D}_{r(A)} \cap \mathbb{R}^n$ s.t. $\mathcal{C} \cap \mathscr{S} \neq \emptyset$. **Then** $\mathcal{C} \subset \mathscr{S}$. **In particular** $\mathcal{C} \subset \mathcal{D}_{r(A)} \setminus \mathcal{D}_{r(A)-1}$.

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Problem statement

Emptiness of spectrahedra

Given $A(x)$ symmetric, with entries in $\mathbb{Q}$, compute a finite set meeting $\mathscr{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$, or establish that $\mathscr{S}$ is empty.

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Real root finding on determinantal varieties

Given any $A(x)$ with entries in $\mathbb{Q}$, compute a finite set meeting each connected component of $\mathcal{D}_r \cap \mathbb{R}^n = \{x \in \mathbb{R}^n : \operatorname{rank} A(x) \leq r\}$.

Particular instance of: Sampling real algebraic sets.

Strategy

1. The **Smallest Rank Property** $(\exists \mathcal{C} \subset \mathcal{D}_{r(A)} : \mathcal{C} \subset \mathscr{S})$ allows to reduce:

$$\boxed{\begin{array}{c}\textbf{Sampling/Optimization over}\\ \textbf{One semi-algebraic set}\end{array}} \longrightarrow \boxed{\begin{array}{c}\textbf{Sampling/Optimization over}\\ \textbf{Many algebraic sets}\end{array}}$$

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Sampling determinantal varieties

- Either the empty list iff
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- Or $(q, q_1, \ldots, q_n) \subset \mathbb{Q}[t]$ s.t.
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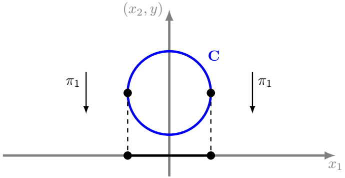
Incidence varieties and critical points

<u>**1st step**</u> *Lifting of the determinantal variety*:

$$A(x)\,Y(y) = A(x)\begin{bmatrix} y_{1,1} & \cdots & y_{1,m-r} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,m-r} \end{bmatrix} = 0.$$

$$U\,Y(y) = I_{m-r}$$



If A is generic, the lifted algebraic set $\mathcal{V}_r$ is **smooth** and **equidimensional**

<u>**2nd step**</u> *Compute critical points* of the map $\pi(x, y) = a_1x_1 + \cdots + a_nx_n$ on $\mathcal{V}_r$:

When a_1 .. a_n are generic, there are **finitely many** critical points.

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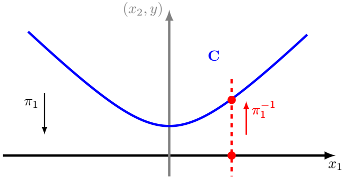

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Computing critical points on incidence varieties

$$A(X)Y = \mathbf{0},\; UY = I_{m-r}$$

↓

$$F_1(X, Y) = \cdots = F_{m(m-r)}(Y) = \cdots = F_{m(m-r)+(m-r)^2} = 0$$

↓

Lagrange System $\mathbf{L}.\mathrm{jac}(\mathbf{F}, [\mathbf{X}, \mathbf{Y}]) = [\mathbf{a}\ \mathbf{0}]$

Computing critical points on incidence varieties

$F_1(X,$

Multi-Linear System

- $\mathbf{F}(\mathbf{X}, \mathbf{Y}) = 0, \mathbf{G}(\mathbf{X}, \mathbf{L}) = 0, \mathbf{H}(\mathbf{Y}, \mathbf{L}) = 0$
- All equations have multi-degree $(1, 1, 0)$ or $(1, 0, 1)$ or $(0, 1, 1)$
- Impact on **multi-linear/sparsity structure** on the number of solutions

Multi-linear Bézout bounds

- Symbolic-homotopy algorithms
  $\rightsquigarrow$ cubic complexity in these bounds

Symbolic Newton iteration
(Hensel lifting)

Jeronimo/Matera/Solerno/Waissbein

Complexity bounds

Complexity for Sampling determinantal varieties

$$\mathcal{O}^{\sim}\left((n + m^2 - r^2)^7 \binom{n + m(m-r)}{n}^6 \right)$$

Complexity for Emptiness of spectrahedra

$$\mathcal{O}^{\sim}\left(n \sum_{r \leq r(A)} \binom{m}{r} (n + p_r + r(m-r))^7 \binom{p_r + n}{n}^6 \right)$$

$\mathcal{O}^{\sim}(k) = \mathcal{O}(k \log^c k)\ \exists\, c \in \mathbb{N}$ with $p_r = (m-r)(m+r+1)/2$.

Remarkable aspects:

Explicit constants in the exponent

When m is fixed, polynomial in n

Strictly depends on $r(A)$

SPECTRA: a library for real algebraic geometry and optimization

What is SPECTRA?

- A MAPLE library, freely distributed
- Depends on **Faugère**'s FGB for computations with Gröbner bases
- Addressed to researchers in Optimization, Convex alg. geom., Symb. comp.

(m, r, n)	RAGLIB	SPECTRA	deg
$(3, 2, 8)$	109	18	39
$(3, 2, 9)$	230	20	39
$(4, 2, 5)$	12.2	26	100
$(4, 2, 6)$	∞	593	276
$(4, 2, 7)$	∞	6684	532
$(4, 2, 8)$	∞	42868	818
$(4, 2, 9)$	∞	120801	1074
$(4, 3, 10)$	∞	303	284
$(4, 3, 11)$	∞	377	284
$(5, 2, 9)$	∞	903	175
$(6, 5, 4)$	∞	8643	726

- RAGLIB = Real algebraic geometry library
- SPECTRA = new algorithms
- deg = degree of Rational Parametrization
- Time in seconds
- ∞ = more than 2 days

Download a beta version: `homepages.laas.fr/snaldi/software.html`

Scheiderer's spectrahedron

$$f = u_1^4 + u_1 u_2^3 + u_2^4 - 3u_1^2 u_2 u_3 - 4u_1 u_2^2 u_3 + 2u_1^2 u_3^2 + u_1 u_3^3 + u_2 u_3^3 + u_3^4$$

One can write $f = v' A(x) v$ with $v = [u_1^2, u_1 u_2, u_2^2, u_1 u_3, u_2 u_3, u_3^2]$

$$A(x) = \begin{bmatrix} 1 & 0 & x_1 & 0 & -3/2 - x_2 & x_3 \\ 0 & -2x_1 & 1/2 & x_2 & -2 - x_4 & -x_5 \\ x_1 & 1/2 & 1 & x_4 & 0 & x_6 \\ 0 & x_2 & x_4 & -2x_3 + 2 & x_5 & 1/2 \\ -3/2 - x_2 & -2 - x_4 & 0 & x_5 & -2x_6 & 1/2 \\ x_3 & -x_5 & x_6 & 1/2 & 1/2 & 1 \end{bmatrix}$$

What information can be extracted?

- No matrices of rank 1 s.t. $A(x) \succeq 0 \longrightarrow f \neq g^2$
- Two matrices of rank 2 s.t. $A(x) \succeq 0 \longrightarrow f = g_1^2 + g_2^2 = g_3^2 + g_4^2$
- No matrices of rank 3 s.t. $A(x) \succeq 0 \longrightarrow f \neq h_1^2 + h_2^2 + h_3^2$

Perspectives

1. **Remove genericity assumptions on the input linear matrix A**

2. **Use of numerical homotopy for studying incidence varieties**

3. **Theoretical toolbox for analyzing singularities of determinantal varieties**

   Surprising applications in optimal control techniques for the contrast imaging problem in medical imagery

   joint work with B. Bonnard, J.-C. Faugère, A. Jacquemard, T. Verron.