A Semi-Lagrangian scheme for a regularized version of the Hughes model for pedestrian flow

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Motivations

- Pedestrian dynamics - *Complex behaviors*
- Micro-meso-macro models - *Interaction rules*
- Basic issue for *crowd management*

Example: *Sziget Festival* in Budapest

Hughes' Model

Hughes model: assumptions

In 2002 R. Hughes proposed a *macropscopic model* for pedestrian dynamics, based on:

1. The total number of individuals is conserved in time and the speed of individuals is link to the density of the surrounding pedestrian flow.
2. Individuals have a common goal, e.g. reaching a certain location in space.
3. Pedestrians want to minimize their estimated travel time but try to avoid regions of high density.

Hughes model: original formulation

Those considerations lead to the following PDE system:

$$\begin{cases} \partial_t m(x,t) - \operatorname{div}(m(x,t)\, f^2(m(x,t))\nabla u(x,t)) = 0, \\ |\nabla u(x,t)| = \dfrac{1}{f(m(x,t))}, \end{cases}$$

where $x \in \Omega$ is the position in space, $t \in (0,T]$ the time, $m = m(x,t)$ the pedestrian density, u the weighted shortest distance to a target.
An example for f is

$$f(m) = 1 - m.$$

Hughes model: a regularization

We consider a regularized version in a d-dimentional closed domain ($d \in \mathbb{N}^+$)

$$\begin{cases} \partial_t m(x,t) - \varepsilon \Delta m(x,t) - \operatorname{div}(m(x,t)\, f^2(m(x,t)) \nabla u(x,t)) = 0, \\ -\varepsilon \Delta u(x,t) + \frac{1}{2} |\nabla u(x,t)|^2 = \dfrac{1}{2f^2(m(x,t)) + \delta}. \end{cases}$$

for small parameters ε and δ in $\mathbb{R}^+$.

Hughes model: boundary conditions

Possible choices for the pedestrian density m at the exit are:

- a given fixed outflow (Neumann BC)
- an outflux which depends on the density (Robin BC)
- or a prescribed pedestrian density (Dirichlet BC).

We choose: let $\mathcal{T}$ denote the common *target/goal* of the crowd

$$\begin{cases} m(x,0) = m_0(t), & \text{on } \Omega \times \{0\}, \\ m(x,t) = 0, & \text{on } \mathcal{T} \times (0,T), \\ u(x,t) = 0, & \text{on } \mathcal{T} \times (0,T), \\ u(x,t) = g(x) & \text{on } \partial\Omega \setminus \mathcal{T} \times (0,T), \\ (\varepsilon \nabla m + f^2(m) \nabla u\, m)(x,t) \cdot \hat{n}(x) = 0, & \text{on } \partial\Omega \setminus \mathcal{T} \times (0,T), \end{cases}$$

where $\hat{n}$ denotes the outer normal vector to the boundary.

Trajectory interpretation

Trajectory interpretation: HJ equation

Given a process α ($\mathcal{F}_s$-measurable for all s and admissible), and $x \in \overline{\Omega}$, we define

$$y_{x,\alpha}(s) = x + \int_0^s \alpha(r)\mathrm{d}r + \sqrt{2\varepsilon}W(s) \quad \text{for all} \quad s > 0,$$

$$\text{and} \quad \tau_{x,\alpha} := \inf\{s > 0\,;\; y_{x,\alpha}(s) \in \partial\Omega\},$$

where W is a d-dimensional Brownian motion adapted to $\mathbb{F}$.
The function $u(x,t)$ defined as

$$u(x,t) = \inf_{\alpha} \left\{ \mathbb{E}\left(\int_0^{\tau_{x,\alpha}} \left[\frac{1}{2}|\alpha(s)|^2 + (2f^2(m(y_{x,\alpha}(s),t)) + \delta)^{-1} \right] \mathrm{d}s \right.\right.$$
$$\left.\left. + g(y_{x,\alpha}(\tau_{x,\alpha})) \right) \right\},$$

is the solution of the HJ equation associated.

Trajectory interpretation: FP equation

Let us consider the Stochastic Differential Equation (SDE)

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t), \mu(X(t), t), t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W(t), \quad \text{for all } t \geq 0, \\ X(0) &= X^0, \end{aligned}$$

where $b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^d$ is a vector-valued function, X^0 is a random vector in $\mathbb{R}^d$, independent of the Brownian motion $W(\cdot)$, with density m_0, and $\mu(\cdot, t)$ is the density of $X(t)$. It can be shown that the eq. above admits a unique solution and that μ is the unique classical solution of the nonlinear FP equation

$$\begin{aligned} \partial\mu - \varepsilon\Delta\mu + \mathrm{div}(b(x, \mu, t)\mu) = 0 \quad &\text{in} \quad \mathbb{R}^d \times [0, \infty[, \\ \mu(\cdot, 0) = m_0(\cdot) \quad &\text{in} \quad \mathbb{R}^d. \end{aligned}$$

Approximate the system

Discrete space

Let us suppose $\Omega = (0, L)^d$.

Given $\Delta t > 0$ and $\Delta x > 0$, let us set $(x_i, t_k) := (i\Delta x, k\Delta t)$, where $i \in \{0, \ldots, M\}^d$ and $k = 0, \ldots, N$, and for a given $A \subseteq \Omega$ set $\mathcal{G}_{\Delta x}(A) := \{i \in \{0, \ldots, M\}^d \; : \; x_i \in A\}$.

We call $B(\mathcal{G}_{\Delta x}(A))$ and $B(\mathcal{G}_{\Delta x, \Delta t}(A))$ the spaces of *grid functions* defined respectively on $\{x_i : i \in \mathcal{G}_{\Delta x}(A)\}$ and $\{(x_i, t_k), i \in \mathcal{G}_{\Delta x}(A), k = 0, \ldots, N\}$.

Discretizing the HJB equation

We define the following linear interpolation operator on $\overline{\Omega}$

$$I[u](\cdot) := \sum_{i \in \mathcal{G}_{\Delta x}(\overline{\Omega})} u(x_i)\beta_i(\cdot) \text{ for } u \in B(\mathcal{G}_{\Delta x}(\overline{\Omega})).$$

We approximate $y_{x,\alpha}(h)$ by $x - h\alpha + \sqrt{\varepsilon h d} Z$, where Z is a random vector in $\mathbb{R}^d$ so then: defining

$$h^{\ell,\pm}_{x,\alpha} := \inf\{\gamma > 0 \; ; \; x - \gamma\alpha \pm \sqrt{\varepsilon\gamma d}\, \mathbf{e}_\ell \in \partial\Omega\} \wedge h,$$

$$y^{\ell,\pm}_{x,\alpha} := x - h^{\ell,\pm}_{x,\alpha}\alpha \pm \sqrt{\varepsilon h^{\ell,\pm}_{x,\alpha} d}\, \mathbf{e}_\ell,$$

A SL-scheme for the HJB equation

For $v \in \mathcal{B}(\mathcal{G}_{\Delta x}(\overline{\Omega}))$ define

$$W(v,i) := \min_{\alpha \in \mathcal{A}} \left\{ \frac{1}{2d} \sum_{\ell=1}^{d} \left[I[v](y_{i,\alpha}^{\ell,\pm}) + \frac{\hat{h}_{i,\alpha}^{\ell,\pm}}{2} |\alpha|^2 + \hat{h}_{i,\alpha}^{\ell,\pm} F(x_i,t) \right] \right\}.$$

where $F(x,t) := 1/(2f^2(m(x,t)) + \delta)$. Find $u \in B(\mathcal{G}_{\Delta x}(\overline{\Omega}))$ such that

$$\begin{cases} u_i = W(u,i) & \text{for all } i \in \mathcal{G}_{\Delta x}(\Omega), \\ u_i = g(x_i) & \text{for all } i \in \mathcal{G}_{\Delta x}(\partial\Omega). \end{cases}$$

Some hints for a fast resolution

Note that, alternatively, the previous problem can be written in the form:

Find $u \in B(\mathcal{G}_{\Delta x}(\overline{\Omega}))$ such that

$$0 = \sup_{\alpha \in \mathcal{A}} \{(B^{\alpha} u)_i - c(\alpha)_i\} \qquad \forall\, i \in \mathcal{G}_{\Delta x}(\overline{\Omega}),$$

where, if $i \in \mathcal{G}_{\Delta x}(\Omega)$,

$$(B^{\alpha} v)_i = v_i - \frac{1}{2d} \sum_{j \in \mathcal{G}_{\Delta x}(\overline{\Omega}),\ \ell = 1,\dots,d} \left[\beta_j(y_{i,\alpha}^{\ell,+}) + \beta_j(y_{i,\alpha}^{\ell,-})\right] v_j,$$

$$c(\alpha)_i = \frac{1}{2d} \sum_{\ell=1}^{d} \left[\frac{1}{2} h_{x_i,\alpha}^{\ell,\pm} |\alpha|^2 + h_{x_i,\alpha}^{\ell,\pm} p(x_i, \mu)\right],$$

This suggests a *policy iteration method*.

Policy iteration method: fundamentals

Lemma

The previous problem admits a unique solution $u^{\Delta x,h}[\mu]$. *Moreover, the sequence, for an* $alpha^0$ *arbitrary in* $\mathcal{A}$,

$$v^k = (B^{\alpha^{k-1}})^{-1} c(\alpha^{k-1}),$$
$$\alpha^k \in \mathrm{argmax}_{\alpha\in\mathcal{A}} \left\{ B^\alpha v^k - c(\alpha) \right\}, \quad k \geq 1,$$

is well-defined. Moreover for all $i \in \mathcal{G}_{\Delta x}(\overline{\Omega})$, *the sequence* v_i^k *is non-increasing, converging to* $u_i^{\Delta x,h}[\mu]$, *and for every limit point* $\alpha^{\Delta x,h}[\mu]$ *of* α^k *we have*

$$0 = (B^{\alpha^{\Delta x,h}[\mu]} u^{\Delta x,h}[\mu])_i - c(\alpha^{\Delta x,h}[\mu])_i \quad \forall\, i \in \mathcal{G}_{\Delta x}(\overline{\Omega}).$$

Discretization of the Fokker-Plank equation

We define

$$E_i = [x_i^1 - \tfrac{1}{2}\Delta x, x_i^1 + \tfrac{1}{2}\Delta x] \times \ldots \times [x_i^d - \tfrac{1}{2}\Delta x, x_i^d + \tfrac{1}{2}\Delta x],$$
$$m_{i,k} := \tfrac{1}{(\Delta x)^d} \int_{E_i} m(x, t_k)\mathrm{d}x.$$

$j \in \mathbb{Z}^d$, $k = 0, \ldots, N-1$ and $\ell = 1, \ldots, d$.
We define also

$$\Phi_{j,k}^{\ell,\pm}[\mu] := x_j + \Delta t\, b(x_j, \mu, t_k) \pm \sqrt{2d\varepsilon\Delta t}\mathbf{e}_\ell.$$

SL-scheme for FP equation

Given $i \in \mathbb{Z}^d$ setting $\phi = \beta_i$ we have the following explicit scheme for $m_{i,k}$:

$$\begin{aligned} m_{i,k+1} &= G(m_k, i, k) \quad \forall\, k = 0, \dots, N-1, \;\; i \in \mathbb{Z}^d, \\ m_{i,0} &= \frac{\int_{E_i} m_0(x)\mathrm{d}x}{(\Delta x)^d} \quad \forall i \in \mathbb{Z}^d, \end{aligned}$$

in which the nonlinear operator G is defined by

$$G(w, i, k) := \frac{1}{2d} \sum_{j \in \mathbb{Z}^d} \sum_{\ell=1}^{d} \left(\beta_i \left(\Phi_{j,k}^{\ell,+} [w_j] \right) + \beta_i \left(\Phi_{j,k}^{\ell,-} [w_j] \right) \right) w_j,$$

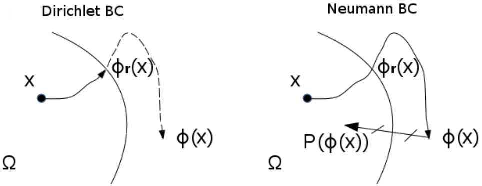
for every $w \in B(\mathbb{Z}^d)$.

Proposition (Weak consistency - in the full space)

Assume that $m : \mathbb{R}^d \times [0,T] \to \mathbb{R}_+$ *is regular enough. Then, assuming that* b *is Lipschitz, for every* $\phi \in C_0^\infty\left(\mathbb{R}^d\right)$ *and* $k = 0, \ldots, N$

$$\int_{\mathbb{R}^d} \phi(x) G_{\Delta x, \Delta t}(m_k, x, t_k) \mathrm{d}x = \int_{\mathbb{R}^d} \phi(x) m(x, t_k) \mathrm{d}x$$
$$+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} b(x, m(t,x), t) \cdot \nabla\phi(x) m(x,t) \mathrm{d}x\mathrm{d}t$$
$$+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} \varepsilon \Delta\phi(x) m(x,t) \mathrm{d}x\mathrm{d}t + O(\Delta x + (\Delta t)^2).$$

Boundary conditions



$$(\Phi_r)_i^{\ell,\pm}[b_k,\mu] := \left(x_j + \nu_i^{\pm}\, b(x_j,\mu,t_k) \pm \sqrt{2d\varepsilon\Delta t}\mathbf{e}_\ell\right),$$

$$\nu_i^{\pm} := \min\left\{\max\{h \geq 0 \text{ s.t. } \quad x_i + \Delta t\, b(x_j,\mu,t_k) \pm \sqrt{2d\varepsilon h} \in \mathcal{T}\}, \Delta t\right\}.$$

$$P(\Phi_i^{\ell,\pm})[b_k,\mu] := \mathcal{P}_\Omega\left(x_i + \Delta t\, b(x_j,\mu,t_k) \pm \sqrt{2d\varepsilon\Delta t}e_\ell\right),$$

where $\mathcal{P}_\Omega : \mathbb{R}^d \to \Omega$ is a projection into Ω following

$$\mathcal{P}_\Omega(z) := \begin{cases} z, & \text{if } z \in \overline{\Omega} \\ 2w - z, \qquad w := \operatorname*{argmin}_{w\in\Omega}|z-w|, & \text{if } z \notin \overline{\Omega}. \end{cases}$$

A Semi-Lagrangian scheme for the system.

Given $\Delta x, \Delta t, h$ we define the following discrete scheme for $k = 0, \ldots, N-1$:

$$\begin{cases} v_{i,k} = W_{\Delta x,h}[m^k](v_k, i) & i \in \mathcal{G}_{\Delta x}(\Omega), \\ v_{i,k} = g(x_i), & i \in \mathcal{G}_{\Delta x}(\partial\Omega), \\ m_i^{k+1} = G_{\Delta x,\Delta t}(\nabla_{\Delta x} v^k, m^k, i) & i \in \mathcal{G}_{\Delta x}(\overline{\Omega}), \\ m_{i,0} = \int_{E_i} m_0(x)dx, & i \in \mathcal{G}_{\Delta x}(\overline{\Omega}). \end{cases}$$

where

$$\nabla_{\Delta x} v_i := \frac{1}{2\Delta x}\Big[v(x_i + \Delta x\, e_1) - v(x_i - \Delta x\, e_1), \ldots, v(x_i + \Delta x\, e_d) - v(x_i - \Delta x\, e_d)\Big].$$

Pedestrian Flux Simulations

Test 1: two exits

We model a crowd in a room $\Omega := (0,1)^2$ with the initial distribution

$$m_0 := \begin{cases} k & x \in [1/3, 2/3]^2 \\ 0 & \text{otherwise.} \end{cases}$$

for $k \in \mathcal{R}$. $\varepsilon = 0.001$,

$$f(x,m) := \frac{1}{2}(1 - m(x,t)), \quad \mathcal{T} := \begin{array}{l} \{x \in \overline{\Omega} | (0, [0.13, 0.27])\} \\ \qquad \cup \{x \in \overline{\Omega} | (1, [0.49, 0.51])\}. \end{array}$$

Here tests with $\delta := 10^{-6}$, $\Delta x = \Delta t = h = 0.08$ and $k = 0.7$, $k = 0.9$.

Influence of the regularization parameter ε

We fix $k = 0.7$

ε	t_e	left exit	right exit
0.04	5.08	54.32 %	45.68 %
0.02	4.62	53.72 %	46.27 %
0.01	3.85	53.40 %	46.59 %
0.005	4.00	52.28 %	47.71 %
0.002	4.10	52.17 %	47.82 %
0.001	4.32	51.85 %	48.14 %
0.0005	4.77	51.40 %	48.59 %

Examples: test with $\varepsilon = 0.001$ and $\varepsilon = 0.01$.

Test 2: the utility of the barriers and turnstiles

Domain $\Omega = [0,1]^2 \setminus \Gamma$ where

$$\Gamma := \left\{ x \in [0,1]^2 \text{ s.t. } \min(0.1 - |x - 0.5|, 0.02 - |y - cs|) \geq 0, s \in \mathcal{N} \right\}$$

Here a test with $k = 0.7$, $a = 0.5$, $c = 0.1$ and diffusion $\varepsilon = 0.001$.

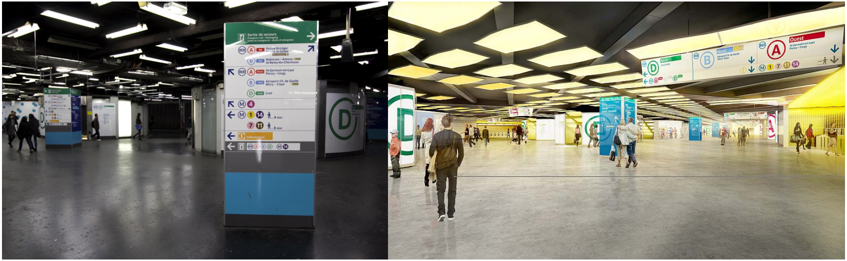
Test 2: compare barriers and diffusion

$\epsilon = \delta_1$	c=0.1	c=0.2	c=0.3	no bar.
0.02	3.54	3.49	3.45	3.42
0.01	3.20	3.02	2.75	2.76
0.005	3.17	2.90	2.70	2.85
0.002	3.17	3.02	3.15	3.32
0.001	3.50	3.65	3.70	3.75
0.0005	3.85	4.12	4.3	4.25
0.0002	3.99	5.75	5.85	5.85
0.0001	6.05	6.65	6.75	6.73

k	c=0.1	c=0.2	c=0.3	no bar.
1.0	5.05	5.25	5.30	5.40
0.9	4.55	4.85	5.12	5.55
0.8	4.15	4.65	4.85	5.15
0.7	3.85	4.12	4.3	4.25
0.6	3.55	3.45	3.42	3.75
0.5	3.35	3.22	3.25	3.21
0.4	2.82	2.75	2.72	2.45

Table : Comparison of the evacuation time varying the parameters of the test.

Test 3: the renovation in “Les Halles” in Paris



“Salle d’exchange RER” of Les Halles, Paris. In the 2014 (left) and in 2016 (project).

Test3: the exchange hall

Evacuation scenario starting from a crowd of pedestrians placed in the center of the hall: simplified example

- no new pedestrians from the platforms below
- no use of the lifts
- $f(x,m) := \frac{1-m(x,t)}{\ell(x)}$ where $\ell = 2$ (turnstiles), 1 (elsewhere).

compare *before/after* renovation.

Test3: Evacuation times from the exchange hall

$$m_0 := \begin{cases} 0.7 & x \in (d_1\frac{1}{3}, d_1\frac{2}{3}) \times (d_2\frac{1}{3}, d_2\frac{2}{3}) \setminus \Gamma \\ 0 & otherwise. \end{cases}$$

where d_1, d_2 are the dimensions of the domain and Γ are the constraints.

k	2014 configuration	2016 configuration	improvement
0.9	9.65	7.22	25.2%
0.8	8.92	6.86	19.8%
0.7	8.13	6.52	22.3%
0.6	7.27	6.18	14.9%
0.5	6.35	5.82	8.3%
0.4	5.39	5.02	6.8%

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Thank you!