A demographic prisonner's dilemma

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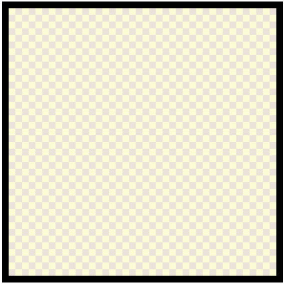
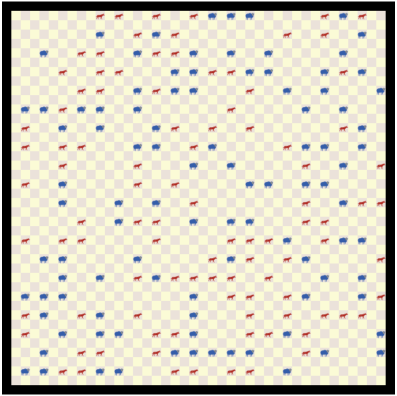
Journées MODE

The Torus



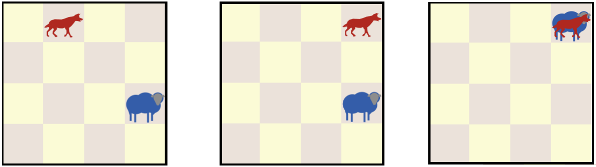
Let $((\mathbb{Z}/m\mathbb{Z})^2)$ be a fixed torus ($m \in \mathbb{N}^*$)

Particles



N particles on the torus.

Move of the player



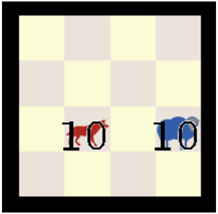
Movement

Particles move following continuous independent symmetric simple random walks of rate $d > 0$.

Evolution

Wealth

Each particle carries a wealth w.
If $w = 0$, the particle dies.



Definition

A configuration σ is an element of $((\mathbb{Z}/m\mathbb{Z})^2) \times \{Red, Blue\} \times \mathbb{N})^N$.
A particle system $(\sigma_t)_t$ is a process taking values in the space of configurations.

Game and Effect

The payoff matrices are with $T > R > 0$ and $S > P > 0$:

$$\begin{pmatrix} (R, R) & (-S, T) \\ (T, -S) & \frac{1}{2}(-2P, 0) + \frac{1}{2}(0, -2P) \end{pmatrix}$$

Every player has a unique action :

- the blue players only cooperate,
- the red players only defect.

Interactions

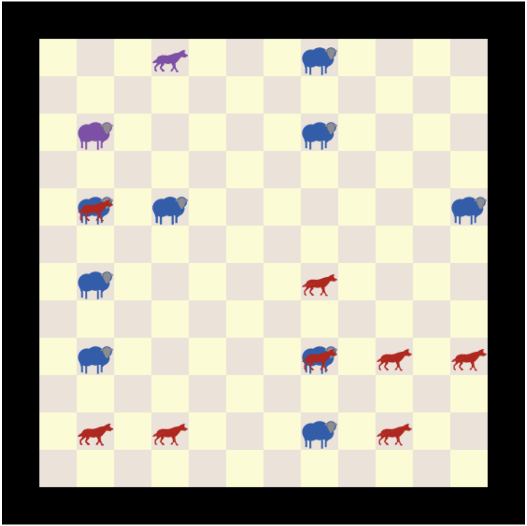
Poisson processes

Couple $(i, j) \leftarrow$ Poisson process independent of everything of parameter $v > 0$.

Spatial condition

Only particles on the same site can play together.

Simulation on Netlogo



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Results

Payoff Matrix

The payoff matrices are with $T > R > 0$ and $S > P > 0$:

$$\begin{pmatrix} (R,R) & (-S,T) \\ (T,-S) & \frac{1}{2}(-2P,0)+\frac{1}{2}(0,-2P) \end{pmatrix}$$

Theorem

There exists a constant $\mu > 0$ depending only on v, d, N and m such that if :

$$\mu R < S$$

then for each initial configuration :
The cooperators will die almost surely.

Payoff Matrix

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Theorem

There exists a constant $\nu > 0$ depending only on d, v, N and m such that if :

$$\nu S < R$$

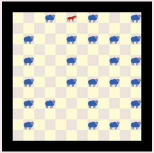
then for each initial configuration :

$$\mathbb{P}(\{\textit{the cooperators live ad vitam eternam}\}) > 0$$

Sketch of the first theorem proof

Context

Only one red particle and blue particles.



Objective

Let W_t^{tot} the sum of the wealth of the blue particles at time t.
Showing that :

$$W_t^{tot} \underset{t \to +\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

Usefull notation for the proof

$\tau(\sigma) = \inf\{n \geq 0,\ t_n$ is the realization of a game Poisson process between a blue player and the red player on the same site $\}$. $\tau(\sigma)$ doesn't depend on the wealth of the players.

$p(\sigma) =$ probability that this game happens in less than $2m+1$ realizations of Poisson processes going from a configuration σ. $p(\sigma)$ doesn't depend on the wealth of the players.

$p = \min_{\sigma} p(\sigma)$

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sketch of the proof of the second theorem

Theorem

There exists a constant $\mu_1 > 0$ depending only on $\lambda_b, \lambda_m, \lambda_g, N$ and m such that if :

$$\mu_1 S < R$$

then for each initial configuration :

$$\mathbb{P}(\{\textit{the cooperators live ad vitam eternam}\}) > 0$$

Idea

The main idea is to consider a ghost system, such that in it, the player don't die but can have negative wealth.
In this system, with the first theorem and with other hypothesis $W_t^{tot} \to +\infty$ a.s. when $t \to +\infty$.

Extension : Birth

Data

- $w_c > 0$: necessary amount of wealth to give birth,
- $w_0 > 0$: initial amount of wealth of the babies.

Poisson process

$i \leftarrow$ Poisson process of parameter b.
$w > w_c \Rightarrow$ Birth Wealth of the parent $w \leftarrow w - w_0$.
Initial wealth of child : w_0.

Mean Field

Mean field assumption

- There is an infinite number of players. ρ (resp β) initial density of red (resp blue) particles
- All the particles have independent laws.
- All the red particle wealths have the same laws (law of a process $(R_t)_t$)
- All the blue particles wealths have the same laws (law of a process $(B_t)_t$).

Spatial approximation

At time $t > 0$:
Particles play against a red particle with probability $\rho\mathbb{P}(R_t > 0)$.
Particles play against a blue particle with probability $\beta\mathbb{P}(B_t > 0)$.

We call the induced stochastic process : $(\sigma_t^{mf})_t$.

Mean Field Result

Theorem

If each particle has an initial wealth of $q_0 > 0$ then : Let $\eta > 1$ satisfying

$$q_0 - \frac{\eta^2 v(C^2 + S^2)}{4(\beta(1 - \frac{1}{\eta^2})C - \rho S)} > 0$$

called the starter condition and also satisfying :

$$\beta C - \rho S > C/\eta^2.$$

Then we have : $\forall t > 0$

$$\mathbb{P}(B_t > 0) \geq 1 - \frac{1}{\eta^2} > 0$$

Example

For $q_0 = 50, v = 1, S = 2, R = 1, \beta = 0.6, \rho = 0.2$ we have a density of blue player always higher than 52%.

Convergence to the Mean Field Model

Intermediate Model

Model

- Finite number of players.
- All the players are on the same site.
- Spatial condition replaced by :
  Game cancelled with probability : $1 - (1/m)^2$

We call the induced stochastic process : $(\hat{\sigma}_t)_t$

Convergence to the Intermediate model

Theorem

Let μ^d be the law of the wealth of all particles in the spatialized model. Let μ be the law of the wealth of all particles in the intermediate model. We have the following convergence :

$$\mu^d \underset{d \to +\infty}{\longrightarrow} \mu$$

Other way of moving : Instinctive move with curiosity

Way of moving

Let $p > 0$,

- With probability p : the particle move randomly
- With probability $1 - p$: the particle move instinctively *i.e.*
  If its last encounter is with a blue particle it stays else it moves.

Theorem

The two first theorem (almost sure extinction and ad vitam eternam survival) hold.

Instinctive moving : Simulation

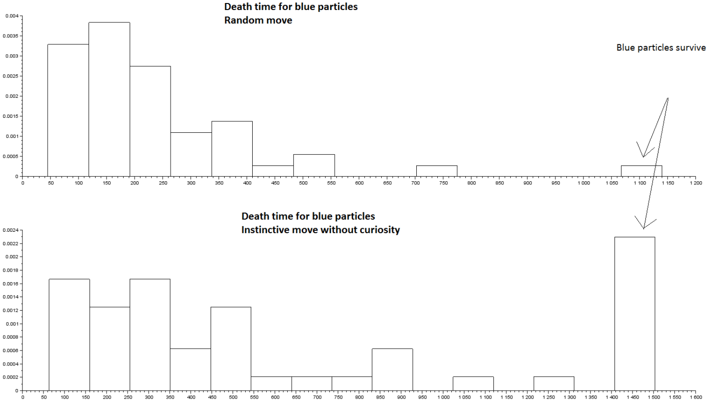

FIGURE: Comparison survival with instinctive moving and with random moving

Thank you for your attention