Generalized Additive Independence models and k-ary capacities in multicriteria decision making

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- Aim of the talk: relate the GAI model with k-ary capacities.

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- Assumption 2: Boundaries:

$$U(x_i^\top, \dots, x_n^\top) = 1, \quad U(x_i^\perp, \dots, x_n^\perp) = 0$$

with $x_i^\top, x_i^\perp$ the best and worst elements of X_i according to $\succsim_i$

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- A GAI model is *p-additive* if any set $S \in \mathcal{S}$ satisfies $|S| \leq p$. Hence, a 1-additive GAI model is a classical additive utility model.

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- Here we consider only normalized k-ary capacities

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- Given a GAI model U with discrete attributes, we define $v : \{0, \ldots, k\}^N \to \mathbb{R}$ by

$$U(x) =: v(\varphi(x)) \qquad (x \in X)$$

and let $v(z) := v(m_1, \ldots, m_n)$ when $z \in \{0, \ldots, k\}^N \setminus \varphi(X)$.

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- By assumptions 1 and 2 on U, it follows that v is a normalized k-ary capacity on N

- Let $v : 2^N \to \mathbb{R}$ be a capacity. Its *Möbius transform* m^v is the (unique) solution of

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- A k-ary capacity is *(at most) p-additive* if $m^v(z) = 0$ whenever $|\mathrm{supp}(z)| > p$, where
$$\mathrm{supp}(z) = \{i \in N \mid z_i > 0\}$$

p-additive discrete GAI models are p-additive k-ary capacities

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Lemma

Let $k \in \mathbb{N}$ and $p \in \{1, \dots, n\}$. A k-ary game v is p-additive if and only if it has the form

$$v(z) = \sum_{x \in \{0,\dots,k\}^N, |\mathrm{supp}(x)| \leq p} v_x(x \wedge z)$$

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It follows that p-additive discrete GAI models are p-additive k-ary capacities (for some $k \in \mathbb{N}$).

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We answer this question for 2-additive discrete GAI models (and the answer is: Yes!)

Why it is important to solve this problem

- Determining a 2-additive GAI model with $k+1$ elements in each attribute by learning yields an optimization problem with
$$(k+1)\binom{n}{1} + (k+1)^2\binom{n}{2}$$
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monotonicity conditions to satisfy.
- If a decomposition into nonnegative nondecreasing terms is possible, one has only to check monotonicity of each term. Then the number of monotonicity conditions drops to
$$n \times k \times \Big[(n-1)(k+1)+1\Big]$$

Why it is important to solve this problem

Comparison table with $k = 4$:

n	4	6	8	10
$\sharp$ of constraints	2000	75 000	2 500 000	78 125 000
$\sharp$ of constraints with monotone decomposition	256	624	1152	1840

n	12	14	20
$\sharp$ of constraints	2 343 750 000	68 359 375 000	$1.526E + 15$
$\sharp$ of constraints with monotone decomposition	2688	3696	7680

The main result

Theorem

Let us consider a 2-additive discrete GAI model U satisfying assumptions 1 and 2. Then there exist nonnegative and nondecreasing functions $u_i : X_i \to [0,1]$, $i \in N$, $u_{ij} : X_i \times X_j \to [0,1]$, $\{i,j\} \subseteq N$, such that

$$U(x) = \sum_{i \in N} u_i(x_i) + \sum_{\{i,j\} \subseteq N} u_{ij}(x_i, x_j) \qquad (x \in X)$$

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- Let $\mathcal{P}_{k,2}$ be the polytope of all normalized 2-additive k-ary capacities
- We prove that any vertex of $\mathcal{P}_{k,2}$ has support of size at most 2
- Since any $v \in \mathcal{P}_{k,2}$ is a convex combination of vertices of $\mathcal{P}_{k,2}$, which are normalized 2-additive k-ary capacities, the desired result follows.

Vertices of $\mathcal{P}_{k,2}$

Theorem

Let $k \in \mathbb{N}$. The set of extreme points of $\mathcal{P}_{k,2}$, the polytope of normalized 2-additive k-ary capacities, is the set of 0-1-valued 2-additive k-ary capacities.

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Theorem

For every $k \in \mathbb{N}$, the size of the support of any 0-1-valued 2-additive k-ary capacity is at most 2.

Vertices of $\mathcal{P}_{k,2}$: sketch of the proof

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- Step 2: We prove that $A_{k,\cdot}$, the matrix defining $\mathcal{P}_{k,\cdot}$, is totally unimodular
- It follows that the polytope $A_{k,\cdot}x \leq b$ is integer $\forall b$, and so is the polytope $A^m_{k,\cdot}m^v \leq b$ for all b (same in the Möbius transform coordinates). Therefore, $A^m_{k,\cdot}$ is also totally unimodular.

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- Step 1: the set of vertices of $\mathcal{P}_{k,\cdot}$ (normalized k-ary capacities) is the set of 0-1-valued k-ary capacities. Therefore, it remains to prove that any vertex of $\mathcal{P}_{k,2}$ is 0-1-valued.
- Step 2: We prove that $A_{k,\cdot}$, the matrix defining $\mathcal{P}_{k,\cdot}$, is totally unimodular
- It follows that the polytope $A_{k,\cdot}x \leq b$ is integer $\forall b$, and so is the polytope $A^m_{k,\cdot}m^v \leq b$ for all b (same in the Möbius transform coordinates). Therefore, $A^m_{k,\cdot}$ is also totally unimodular.
- As $A^m_{k,2}$ is a submatrix of $A^m_{k,\cdot}$, it is also totally unimodular. Therefore, the vertices of $\mathcal{P}^m_{k,2}$ are integer-valued.

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- We prove that v is 0-1-valued iff m^v is $\{-1,0,1\}$-valued. The desired result then follows.

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Preliminary step: one shows that the vertices of $\mathcal{P}_{k,2}$ with support included in, say, $\{1,2\}$, are in bijection with the antichains (which are of size at most $k+1$) of the lattice $(k+1)^2$.

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Theorem

Let $k \in \mathbb{N}$ and consider the polytope $\mathcal{P}_{k,2}$. The following holds.

1. *For any $i \in N$, the number of vertices with support $\{i\}$ is k.*
2. *For any distinct $i, j \in N$, the number of vertices with support included in $\{i, j\}$ is $\binom{2k+2}{k+1} - 2$.*
3. *The total number of vertices of $\mathcal{P}_{k,2}$ is*

$$\left[\binom{2k+2}{k+1} - 2\right]\frac{n(n-1)}{2} - kn(n-2).$$

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- $\mathrm{supp}(v) \subseteq \{1,2\}$ iff its minimal winning coalitions have their support in $\{1,2\}$, and there can be at most $k+1$ distinct minimal winning coalitions
- Suppose that $\mathrm{supp}(v) \subseteq \{1,2\}$. Denote by $x^1, \ldots, x^q$ the minimal winning coalitions of v, arranged such that $x^1_1 < x^2_1 \cdots < x^q_1$. Then $m^v(x^\ell) = 1$ for all $\ell = 1, \ldots, q$, $m^v(x^\ell \vee x^{\ell+1}) = -1$ for $\ell = 1, \ldots, q-1$, and $m^v(x) = 0$ otherwise.

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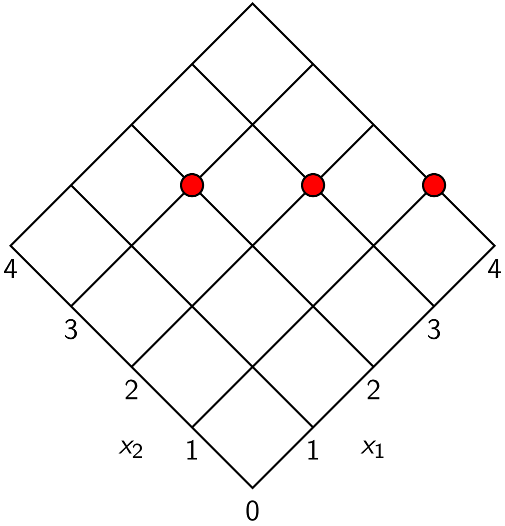

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