Semidefinite approximations of the polynomial abscissa

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The abscissa function

Set $\mathcal{Q} \subseteq \mathbb{R}^n$ compact, semialgebraic.

Here $\mathcal{Q} = [-1, 1]^n$.

Polynomial p parameterized in q:

$$p\colon s \mapsto p(q, s) = \sum_{k=0}^{m} p_k(q) s^k \in \mathbb{R}[s],$$

- $s \in \mathbb{C}$
- parameter $q = (q_1, \ldots, n) \in \mathcal{Q}$
- polynomials $p_k \in \mathbb{R}[q]$
- $p_m \equiv 1,\ m > 0$, i.e. p monic and non-constant

The abscissa function

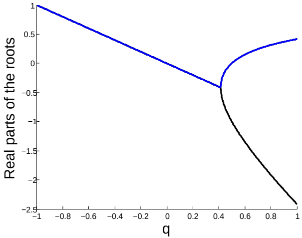
Zeros of $p(q,\cdot)$: $s_r(q),\ r = 1,\dots,m.$

Abscissa of p:

$$a\colon \mathcal{Q} \to \mathbb{R},\ q \mapsto a(q) = \max_{r=1,\dots,m} \Re(s_r(q)).$$

Example:
The damped oscillator

$$p:\ s \mapsto p(q,s) = s^2 + 2qs + 1 - 2q$$



The abscissa is continuous, not Lipschitz-continuous, but Hölder continuous.

The abscissa function

Split s and p in its real and imaginary parts:

$$s = x + iy,$$
$$p(q,s) = p_{\Re}(q,x,y) + ip_{\Im}(q,x,y).$$

Write abscissa as

$$a(q) = \max\{x \in \mathbb{R} : \exists y \in \mathbb{R} : p_{\Re}(q,x,y) = 0 = p\Im(q,x,y)\}.$$

Example: The damped oscillator $p(q,s) = s^2 + 2qs + 1 - 2q$

$$a(q) = \max\{x \in \mathbb{R} : \exists y \in \mathbb{R} :$$
$$x^2 - y^2 + 2qx + 1 - 2q = 0 = 2xy + 2qy\}$$

The abscissa function

Define the set $\mathcal{Z}$ of zeros of p:

$$\mathcal{Z} = \{(q, x, y) \in \mathcal{Q} \times \mathbb{R}^2 : p_{\Re}(q, x, y) = 0 = p_{\Im}(q, x, y)\}$$

compact, basic closed semi-algebraic.

Write abscissa as

$$a(q) = \max\{x \in \mathbb{R} : \exists y \in \mathbb{R} : (q, x, y) \in \mathcal{Z}\}.$$

Motivation

Where does the abscissa occur?

linear systems control

space of controller parameters

What is already known?

Burke, Lewis, Overton: variational analysis

Cross: survey on the abscissa function and its applications in systems control

Infinite-dimensional linear programming problem

$$\rho = \inf_{v \in \mathscr{C}(\mathcal{Q})} \int_{\mathcal{Q}} v(q)\, dq$$
$$\text{s.t.} \quad v(q) - x \geq 0 \text{ for all } (q, x, y) \in \mathcal{Z}$$

A solution v gives an upper approximation of the abscissa.

Stone-Weierstraß: can replace $\mathscr{C}(\mathcal{Q})$ by $\mathbb{R}[q]$.

$\mathscr{C}(\mathcal{Q})$ ... continuous functions on $\mathcal{Q}$.

Finite-dimensional semidefinite programming problem

$$\rho_d = \inf_{v_d,\sigma_0,\sigma_j,\tau_{\Re},\tau_{\Im}} \int_{\mathcal{Q}} v_d(q)\, dq$$

$$\text{s.t. } v_d(q) - x = \sigma_0(q,x,y) + \sum_{j=1}^{n} \sigma_j(q,x,y)(1-q_j^2)$$
$$+ \tau_{\Re}(q,x,y) p_{\Re}(q,x,y) + \tau_{\Im}(q,x,y) p_{\Im}(q,x,y)$$

- for all $(q,x,y) \in \mathbb{R}^n \times \mathbb{R}^2$
- $v_d \in \mathbb{R}[q]_{2d}$
- $\sigma_0 \in \Sigma[q,x,y]_{2d}$, $\sigma_j \in \Sigma[q,x,y]_{2d-2}$ for $j = 1, \ldots, n$
- $\tau_{\Re}, \tau_{\Im} \in \mathbb{R}[q,x,y]_{2d-m}$

SDP hierarchy

The quadratic module $\mathcal{M} := M(1 - q_j^2, \pm p_{\Re}, \pm p_{\Im})$ is archimedean.

The Lasserre hierarchy converges: $\rho_d \to \rho$.

Explanation:

The SDP is a strengthening of the LP: $\rho_d \geq \rho$.

Also for d increasing: $\rho_{d+1} \leq \rho_d$.

Because: $\{g > 0 \text{ on } \mathcal{Z}\} \subseteq M \subseteq \{g \geq 0 \text{ on } \mathcal{Z}\}$.

Main result

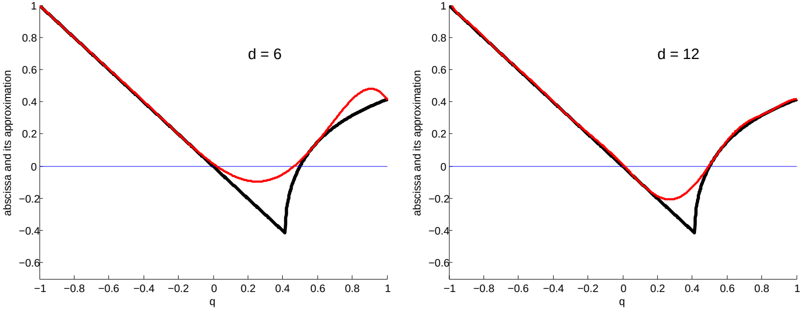
Theorem

Let $v_d \in \mathbb{R}[q]_{2d}$ be near optimal solutions for the SDP hierarchy, i.e. $\int_{\mathcal{Q}} v_d(q)\,dq \leq \rho_d + \frac{1}{d}$, and consider the associated sequence $(v_d)_{d \geq d_0} \subset L^1(\mathcal{Q})$.

Then v_d converges to the abscissa a in L^1 norm on $\mathcal{Q}$.

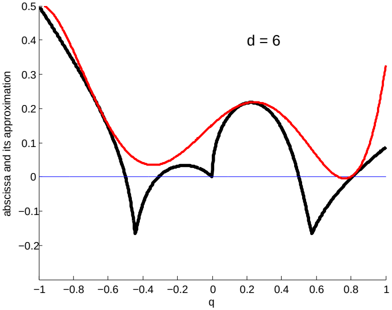
The damped oscillator

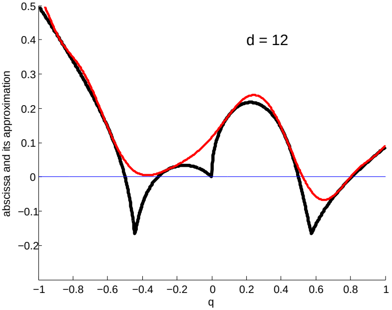
$$p : \; s \mapsto p(q,s) = s^2 + 2qs + 1 - 2q$$



One-dimensional example

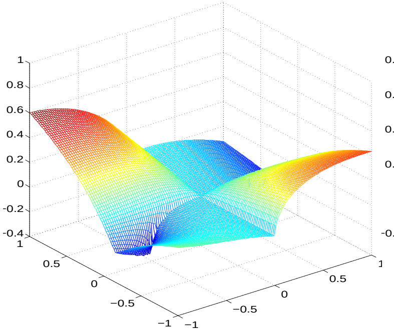
$$p: \; s \mapsto p(q,s) = s^3 + \tfrac{1}{2}s^2 + q^2 s + (q - \tfrac{1}{2})q(q + \tfrac{1}{2})$$



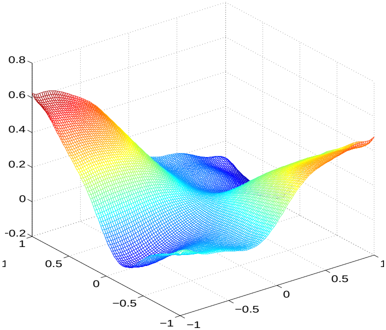


Two-dimensional example

$$p: \; s \mapsto p(q,s) = s^3 + (q_1 + \tfrac{3}{2})s^2 + q_1^2 s + q_1 q_2$$



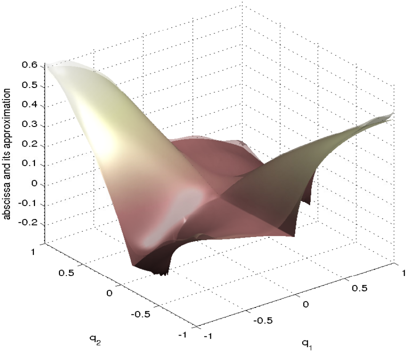
Abscissa

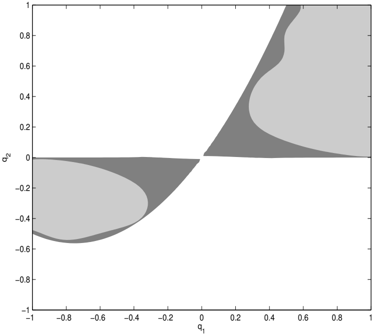


Approximation $d = 10$

Two-dimensional example

$$p: \; s \mapsto p(q,s) = s^3 + (q_1 + \tfrac{3}{2})s^2 + q_1^2 s + q_1 q_2$$





$$d = 10$$

Thank you for your attention!