# A new direction in polynomial time interior-point methods for monotone linear complementarity problem

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# Outline

# Notation

- $M \in \mathbb{R}^{n\times n}$
- $q \in \mathbb{R}^n$
- The Linear Complementarity Problem (LCP) consists in finding vectors $z \geq 0$ and $s \geq 0$ such that

$$\begin{aligned} Mz + q &= s \\ zs &= 0 \end{aligned},$$

  $zs = (z_i s_i)_{1\leq i\leq n}$.
- $(z, s)$ are feasible if they verify $z, s \geq 0$ and $Mz + q = s$.
- $M$ satisfies a monotonicity property : all vectors $z \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ that satisfy $Mz - s = 0$ have $z^T s \geq 0$.

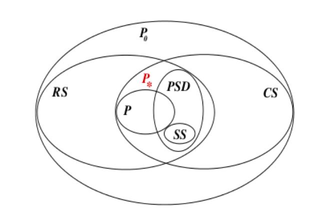
# Linear Complementarity Problem : Applications

## Several applications of LCP :

convex hulls in a plane, nash equilibrium in bimatrix games, absolute value equation, optimality conditions of optimization problems (Linear, Convex Quadratic )...

## Monotoncity property of $M$ :

- P-matrix (unique solution)
- Positive Semi-Definite (feasible => solvable)
- Skew-Symmetric (Linear Programming)



# Interior-Point Method (IPM)

(LCP)

$$
\begin{aligned}
Mz + q &= s \\
zs &= 0 \\
z, s &\geq 0
\end{aligned}
$$

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Given $\mu \geq 0$

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This system has a unique solution $(z(\mu), s(\mu))$ if the Interior Point Condition holds.

**Interior Point Condition :**

$$\exists z^0, s^0 \text{ such that } s^0 = Mz^0 + q,\ \ z^0 > 0,\ \ s^0 > 0$$

$(x(\mu), s(\mu))_\mu$ defines the central path, leading to the optimal solution $(\mu \to 0)$.
IPMs follow the central path approximately.

# The most simple IPM : algorithm

**Data**:

an update parameter $\theta$, $0 < \theta < 1$ ;

**Begin** ;

$z = z^0$, $s = s^0$, $\mu := \mu^0$ ;

**while** $n\mu \geq \epsilon$ **do**

$\quad \mu := (1 - \theta)\mu;$

$\quad (z, s) := (z, s) + (\Delta z, \Delta s);$

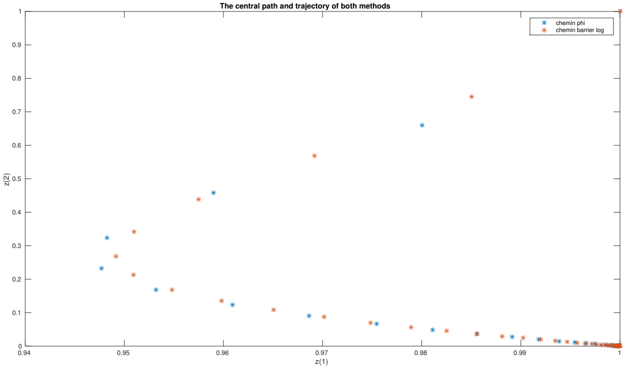
**end**

**Algorithm 1:** Full Newton step IPM

$(\Delta z, \Delta s)$ is the unique solution of the system

$$\begin{cases} M\Delta z = \Delta s \\ z\Delta s + s\Delta z = \mu - zs \end{cases} \tag{1}$$

# The most simple IPM : : illustrations




Complexity for monotone LCP to get $z^T s \leq n\epsilon$ : $\mathcal{O}(\sqrt{n}\log(\frac{n}{\epsilon}))$ where $\theta = \mathcal{O}(\frac{1}{\sqrt{n}})$ has a fixed value.

# IPM : New Directions

Given $\mu \geq 0$,

$$(LCP) \qquad \begin{aligned} Mz + q &= s \\ \varphi(zs) &= \mu \\ z, s &\geq 0 \end{aligned}$$

- Introduced in 03' by Darvay.
- IPM with full Newton step has a complexity in $\mathcal{O}(\sqrt{n}\log(\frac{n}{\epsilon}))$ for $\varphi(.) = \sqrt{.}$

**Warning :**

It is not the same as $zs = \varphi^{-1}(\mu)$ since we take a Newton step.

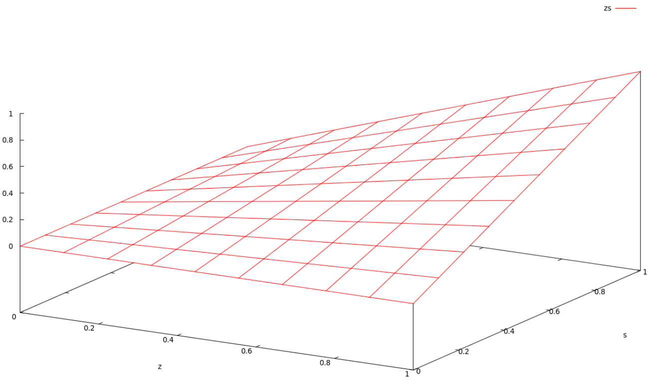



Figure : Level surface of *zs*

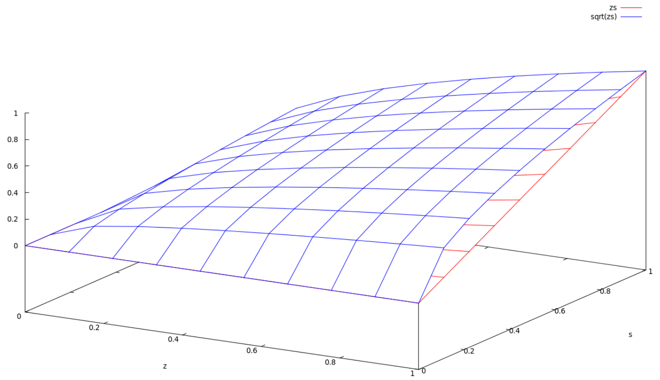



Figure : Level surfaces of $zs$ and $\sqrt{zs}$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\varphi(0) = 0$, $\varphi \in C^2$, concave and invertible.

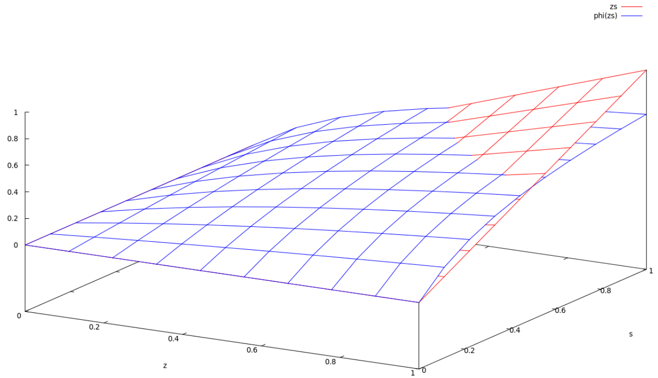



Figure : Level surfaces of $zs$ and $\varphi(zs) = \frac{zs}{zs+0.5}$

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**Theorem :**

Let $\bar{\mu} \geq \mu^0 = \frac{(z^0)^T s^0}{n}$. After at most

$$\mathcal{O}(\sqrt{n} \log(\frac{n}{\epsilon}))$$

iterations, we have $\varphi(zs)^T e \leq n\epsilon$. The algorithm generates a sequence of update parameter $\theta^k$, guarantees feasibility of the iterates and quadratic convergence of the Newton process.

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Concavity of $\varphi$ gives that for $zs$ sufficiently small

$$\varphi(zs) \approx zs\varphi'(0) \tag{2}$$

## Quadratic convergence of the Newton process

$$\delta(z^+ s^+, \mu) \leq \delta(zs, \mu)^2$$

where $\delta(zs, \mu)$ is a proximity measure.

- classical proximity measure :

$$\frac{1}{2} \left\| \frac{zs}{\mu} - \frac{\mu}{zs} \right\|_2$$

- new proximity measure :

$$\frac{1}{2} \left\| \frac{\varphi'(0)}{\varphi'(zs)} \left( \frac{zs}{\mu} - \frac{\mu}{zs} \right) \right\|_2$$

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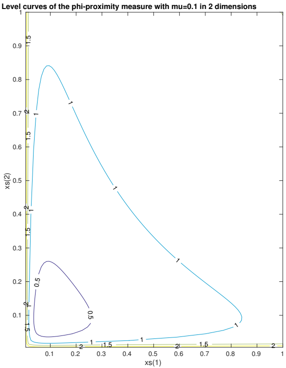
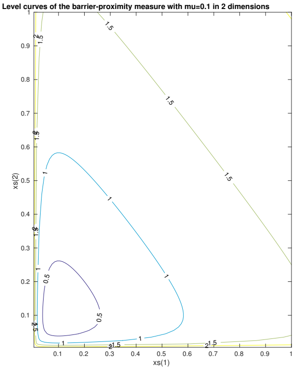


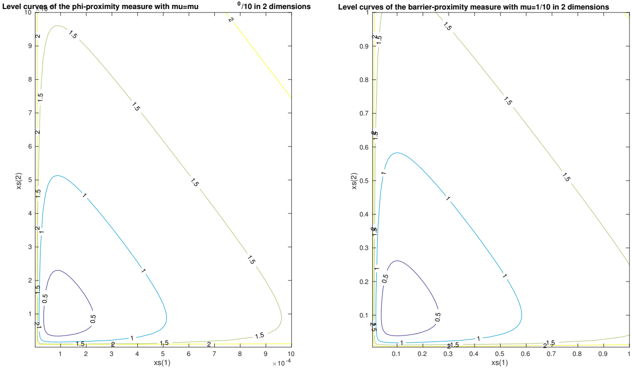


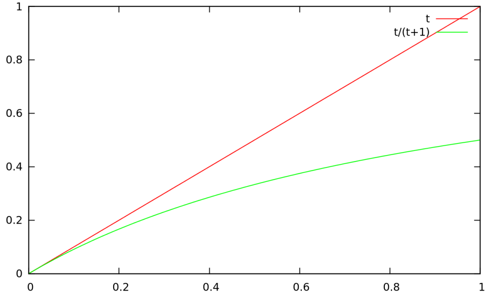

Figure : Level curves for both proximity measures at $\mu^0/\bar{\mu}$ and $\mu^0$

# Behaviour on an example for LO

We will consider $\varphi(t) = \frac{t}{t+1}$ so $\varphi'(t) = \frac{1}{(1+t)^2}$.

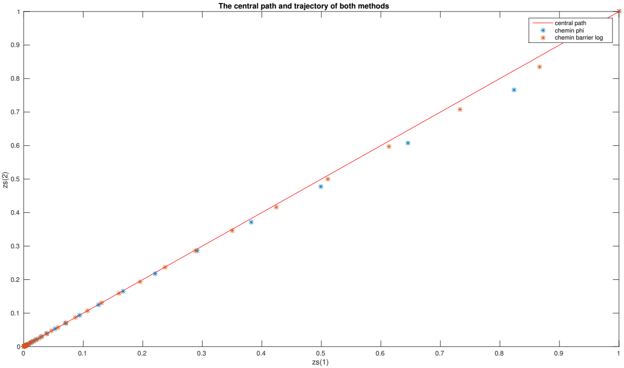
Observations :

- sequence of update parameters
- central path
- domain of quadratic convergence



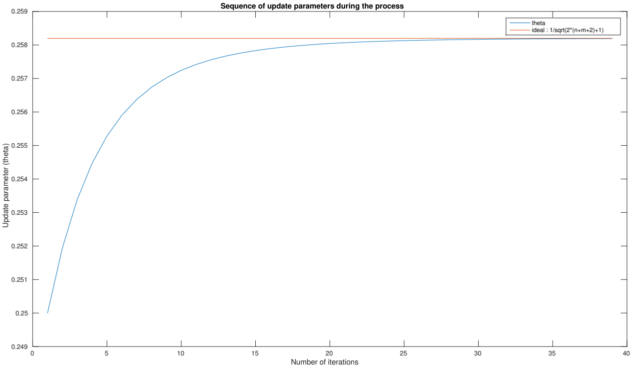
# Behaviour on an example for LO

The central path and the iterates of both methods.
One should note that this figure is presented in the projection of the space of *zs* in $\mathbb{R}^2$.



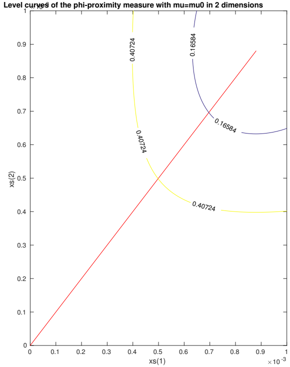
# Behaviour on an example for LO

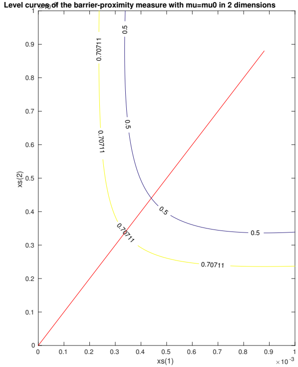
The sequence of update parameter, which converge to its upper bound $\frac{1}{\sqrt{2n+1}}$



# Behaviour on an example for LO

The level curves which guarantees quadratic convergence of the Newton process.



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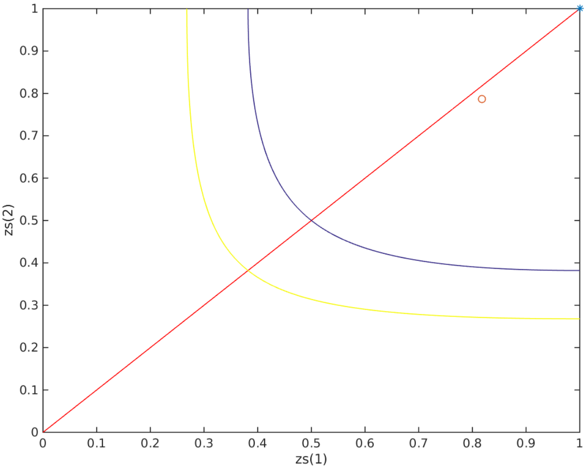


Figure : Step : classical-direction. '*' before Newton, 'o' after Newton.

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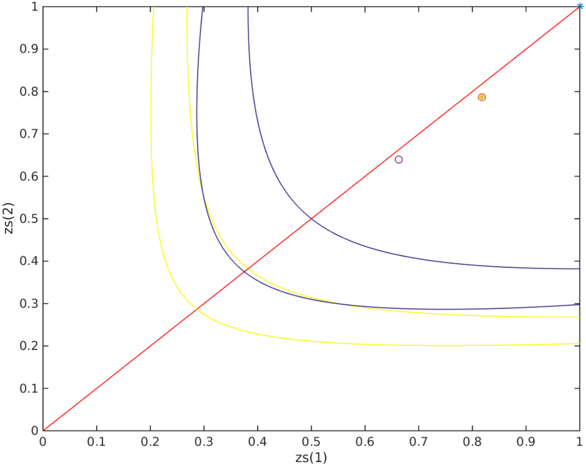


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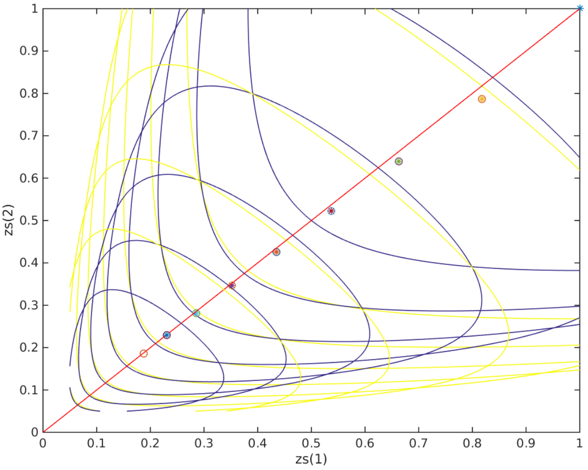


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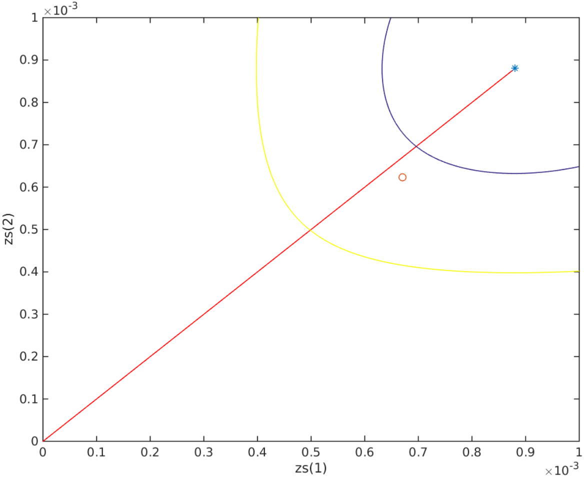


Figure : Step : $\varphi$-direction. '*' before Newton, 'o' after Newton.

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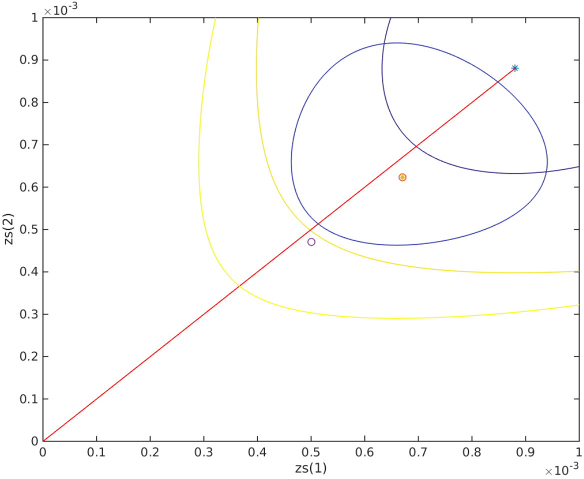


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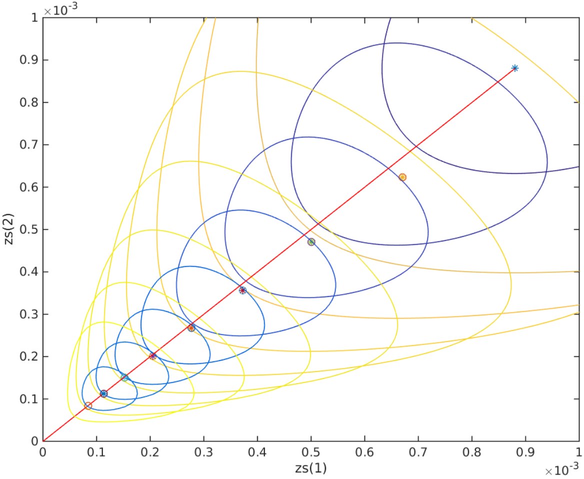


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All the computation with a lower value of $\mu \rightarrow$ linear systems becomes harder to solve ?
Condition number in function of $\mu$ (so for our direction it is $\mu/\mu^0$).

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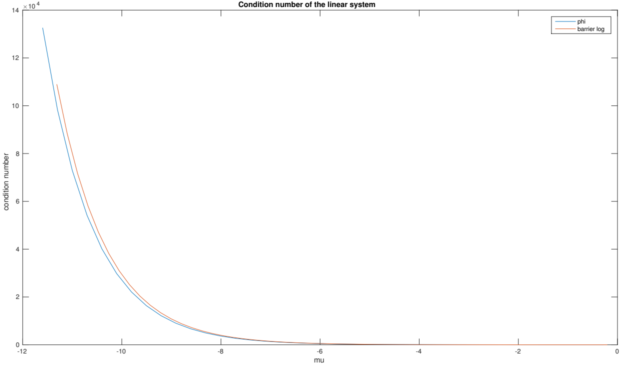


Behaviour of both system is pretty much the same.

# Sum up

An new IPM method with full Newton step :

- different steps
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And now what ?

# What now ? Initialization

## Embedding technique

Larger system with artificial initial point

## Infeasible IPM

$$s - Mz - q = \nu(s^0 - Mz^0 - q),\ z, s \geq 0,\ zs = \mu \ , \qquad (3)$$

asymptotically feasible.
Recent developments :

- improved bound for LO [Roos 15']
- $\varphi(.) = \sqrt{.}$, [Mansouri et al., 14' and 15']

# What now ? Large Update method

## Large Update method

- Small Update : takes all the Newton step => find $\theta$.
- Large Update : choose $\theta$ => takes a damped Newton step.

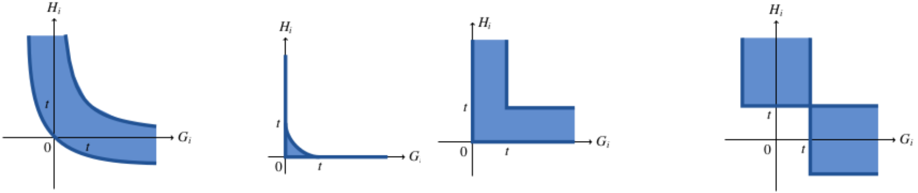
Challenge : "the irony of IPMs"

- Small-update methods : $O(\sqrt{n}\log(\frac{n}{\epsilon}))$ - inefficient in practice
- Large-update methods : $O(n\log(\frac{\sqrt{n}\log(n)}{\epsilon}))$ - very efficient in practice

# What now ? A new hobby, relaxation methods for MPCC

Mathematical Program with Complementarity Constraint (MPCC)

$$
(\text{MPCC}) \begin{cases} \min\limits_{x \in \mathbb{R}^n} f(x) \\ h_i(x) = 0, \;\; i = 1, \ldots m \\ g_i(x) \leq 0, \;\; i = 1, \ldots p \\ 0 \leq G_i(x) \perp H_i(x) \geq 0, \;\; i = 1, \ldots q \end{cases} , \qquad (4)
$$



# Merci de votre attention !

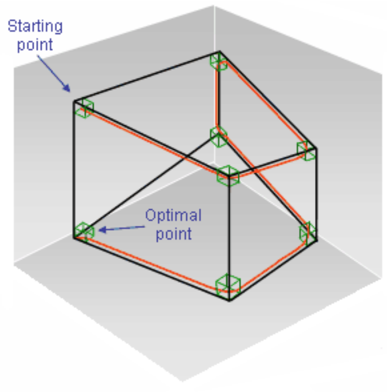



Figure : Central path on the Klee-Minty cube