The Splitting Game: value and optimal strategies

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General context

Game = interdependent strategic interaction between players

- Nature of the interaction
  - Cooperative
  - Evolutionary
  - Non-cooperative
- Number of players
  - Infinitely many (non-atomic)
  - $N > 2$ players
  - 2 players
- Players' preferences
  - Structure (potential)
  - Identical (coordination, mean field, congestion)
  - Opposite (zero-sum)

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  - **Opposite (zero-sum)**

Zero-sum games

A zero-sum game is a triplet (S, T, g), where

- S is the set of actions of player 1
- T is the set of actions of player 2
- $g : S \times T \to \mathbb{R}$ is the payoff function

The game is said to be finite when $S = \Delta(I)$ and $T = \Delta(J)$ are probabilities on finite sets (g is a matrix and actions are mixed strategies)

It admits a value when

$$\sup_{s\in S} \inf_{t\in T} g(s,t) = \inf_{t\in T} \sup_{s\in S} g(s,t)$$

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We are interested in the following two questions:

(a) Existence and description of the value

(b) Existence and description of optimal strategies (or ε-optimal)

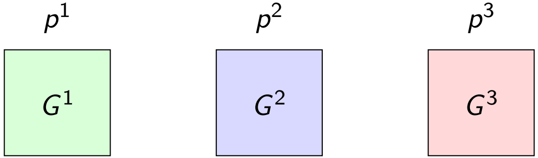
Zero-sum games with incomplete information

– Consider a finite family of matrix games $(G^k)_{k\in K}$, where $G^k = (I, J, g^k)$ corresponds to the state of the world occurring with probability p^k

– The state of the world stands for the player's types, their beliefs about the opponents' types, and so on

– Each player has an information set, i.e. a partition of the state of world

Example: three states and information sets $\{1\},\{2,3\}$ and $\{1,2\},\{3\}$



– A state of the world occurs according to $p \in \Delta(K)$; player 1 knows whether it is $\{1\}$ or $\{2,3\}$, and player 2 knows whether it is $\{1,2\}$ or $\{3\}$

An equivalent formulation

Alternatively, the players' information structure (i.e. the set of states, the information sets and the probability p) can represented as follows:

- The set of possible types is a product set $K \times L$ and the payoff function depend on the pair of types, i.e. $G^{k\ell} : I \times J \to \mathbb{R}$
- $\pi \in \Delta(K \times L)$ is a probability measure on the set of types
- A couple of types (k, ℓ) is drawn according to π. Player 1 is informed of k and player 2 of ℓ

In the previous example: $K = L = \{1, 2\}$ and

$$\pi = \begin{array}{|c|c|} \hline p^1 & p^3 \\ \hline p^2 & 0 \\ \hline \end{array}$$

Remarks. – The players have private, dependent information
– If L is a singleton, the incomplete information is **on one side**

Repeated games with incomplete information

- Aumann and Maschler consider the repetition of games with incomplete information to analyze the strategic use of private information
- A **repeated game with incomplete information** is described by a 6-tuple (I, J, K, L, G, π) where I and J are the sets of actions, K and L the set of types, $G = (G^{k\ell})_{k,\ell}$ the payoff function and π a probability on $K \times L$
- The game is played as follows. First, a couple (k, ℓ) is drawn according to π and each player is informed of one coordinate. Then, the game $G^{k\ell}$ is played over and over: at each stage $m \geq 1$, knowing the past actions, the players choose actions (i_m, j_m)

Strategies and evaluation of the payoff

- Strategies are functions from histories to mixed actions. Here $\sigma = (\sigma_m)_m$ where $\sigma_m : K \times (I \times J)^{m-1} \to \Delta(I)$ and similarly τ stands for strategy of player 2
- Let $\mathbb{P}^{\pi}_{\sigma,\tau}$ be the unique probability distribution on finite plays $h_m = (k, \ell, i_1, j_1, \ldots, i_{m-1}, j_{m-1})$ induced by π, σ and τ
- Player 1 maximizes $\gamma_\theta(\pi, \sigma, \tau) = \mathbb{E}^{\pi}_{\sigma,\tau}[\sum_{m\geq 1} \theta_m G^{k\ell}(i_m, j_m)]$ where $\theta_m \geq 0$ is the weight of stage m
- Two important cases: the n-stage game and the λ-discounted game which correspond to weights:

$$\left(\frac{1}{n}\mathbb{1}_{\{m\leq n\}}\right)_{m\geq 1} \quad \text{and} \quad \left(\lambda(1-\lambda)^{m-1}\right)_{m\geq 1}$$

Approches: Horizon, Value and Strategies

- **Fixed duration** (fixed evaluation θ)
  - (a) ...
  - (b) ...
- **Asymptotic approach** ($\sup_{m\geq 1} \theta_m \to 0$)
  - (a) ...
  - (b) ...
- **Uniform approach** (the weights are "sufficiently small")
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  - (b) ...

Approches: Horizon, Value and Strategies

- **Fixed duration** (fixed evaluation θ)
  - (*a*) Description of the values
  - (*b*) Description of optimal strategies
- **Asymptotic approach** ($\sup_{m\geq 1} \theta_m \to 0$)
  - (*a*) Convergence of the values and caracterization of the limit
  - (*b*) Description of asymptotically optimal strategies
- **Uniform approach** (the weights are "sufficiently small")
  - (*a*) Existence of the uniform value
  - (*b*) Description of robust optimal strategies

Main results on RGII (one or two sides)

Info \ Horizon	Asymptotic	Uniform
One side	$\lim_{\theta \to 0} V_\theta = \mathrm{Cav}\, u$ Aumann - Maschler 67	$V_\infty = \mathrm{Cav}\, u$ Aumann - Maschler 67
Two sides	$\lim_{\theta \to 0} V_\theta = MZ(u)$ Mertens-Zamir 71	V_∞ does not exist

The benefit of private information

The use of private information has two effects during the play

(1) Transmits information about the true types. Indeed, let π_m be the conditional probability on $K \times L$ given h_m under $\mathbb{P}^{\pi}_{\sigma,\tau}$. The players jointly generate the martingale of posteriors $(\pi_m)_m$

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(2) Provides an instantaneous benefit $\Longrightarrow$ irrelevant in the long run:

$$\left| \gamma_\theta(\pi, \sigma, \tau) - \mathbb{E}^{\pi}_{\sigma,\tau}\left[\sum_{m \geq 1} \theta_m u(\pi_m) \right] \right| \leq C \left(\sup_{m \geq 1} \theta_m \right)^{1/2}$$

where $u(\pi)$ is the value of the average game $\sum_{k,\ell} \pi^{k\ell} G^{k\ell}$, i.e.

$$u(\pi) := \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k,\ell} \pi^{k\ell} G^{k\ell}(x, y)$$

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- The previous remark motivated the introduction of **the splitting game** by Sorin and Laraki 2001

The splitting game (one side)

- Consider the case $|L| = 1$ (i.e. player 1 is informed and player 2 is not)
- The initial probability can be seen as $p \in \Delta(K)$ and the possible games as $(G^k)_{k \in K}$
- Let $u(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k \in K} p^k G^k(x, y)$
- $\left| V_\theta(p) - \sup_{(p_m)_{m \geq 1}} \mathbb{E}[\sum_{m \geq 1} \theta_m u(p_m)] \right| \leq C \left(\sup_{m \geq 1} \theta_m \right)^{1/2}$

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Taking the limit, we obtain a martingale optimization problem:

$$V(p) = \sup_{\mathbf{p}\in\mathcal{M}(p)} \mathbb{E}\left[\int_0^1 u(p_t)dt \right]$$

where $\mathcal{M}(p)$ is the set of càdlàg martingales with $\mathbf{p}_{0^-} = p$, a.s.

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What is the value ? What about optimal martingales?
What if there are constraints on the set of admissible martingales?

The splitting game (two sides, independent case)

- In the independent case, $\pi = p \otimes q$, with $p \in \Delta(K)$ and $q \in \Delta(L)$
- The initial probability can be writen as (p, q)
- Let $u(p, q) = \mathrm{val}\left(\sum_{k,\ell} p^k q^\ell G^{k\ell}\right)$
- $|V_\theta(p, q) - W_\theta(p, q)| \leq C\left(\sup_{m\geq 1} \theta_m\right)^{1/2}$ where

$$W_\theta(p, q) = \sup_{(p_m)_m} \inf_{(q_m)_m} \mathbb{E}\left[\sum_{m\geq 1} \theta_m u(p_m, q_m)\right]$$

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What is the value ? What about optimal martingales?

- The independent SG is defined and studied by Laraki 2001

Main results

(1) Existence of the value $W_\lambda(p, q)$

(2) Convergence of $W_\lambda(p, q)$ to $\lim_{\lambda\to 0} V_\lambda = MZ(u)$

(3) Variational characterization of $MZ(u)$

Further results

Two sides SG

- Cardaliaguet, Laraki and Sorin 2011 prove the convergence of $W_\theta(p,q)$ to $MZ(u)$ as $\theta \to 0$

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One side, time-dependent SG

- In the framework of continuous-times games, Cardaliaguet and Rainer 2009 study the splitting game

$$V(t_0, p) = \sup_{\mathbf{p} \in \mathcal{M}(p)} \mathbb{E}\left[\int_{t_0}^{1} u(t, p_t) dt\right]$$

- Characterization of the value and of an optimal martingale

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- Characterization of the value and of an optimal martingale

Two sides, time-dependent SG

- CLS 11 prove the convergence of $W_\theta(t,p,q)$ as $\theta \to 0$ and characterize the limit

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The splitting game: uniform value and optimal strategies

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- Convergence of $V_\theta(\pi)$ to $MZ(u)$ as $\theta \to 0$, for general π
- A comparison principle for the the uniqueness of a solution to MZ
- Existence of the uniform value in the SG
- Exhibition of a couple of optimal strategies with the additional property that the martingale $(\pi_m)_m$ is constant after stage 2

The splitting game

- The splitting game is a stochastic game played on $\Delta(K \times L)$, and where the actions are splittings
- It is described by a 7-tuple $(S, A, B, u, \Phi, \pi, \theta)$ where
  - $S = \Delta(K \times L)$ is the set of states
  - A and B are the sets of splittings
  - $u : S \to \mathbb{R}$ is the payoff function
  - $\Phi : S \times A \times B \to \Delta(S)$ is the transition function
  - $\pi \in S$ is the initial state
  - $\theta = (\theta_m)_m$ is the sequence of weights for the stages
- Strategies are functions from finite histories into splittings
- Player 1 maximizes $\mathbb{E}^{\pi}_{\sigma,\tau}[\sum_{m\geq 1} \theta_m u(\pi_m)]$ where $\mathbb{P}^{\pi}_{\sigma,\tau}$ is the unique probability distributions on finite histories induced by π, σ, τ
- We denote the maxmin and minmax by $W^-_\theta(\pi)$ and $W^+_\theta(\pi)$

The splittings

- For any $\pi \in \Delta(K \times L)$ let
  - Let $\pi^K \in \Delta(K)$ be its marginal on K
  - Let $\pi^{L|K} \in \Delta(L)^K$ be the matrix of conditionals on L given $k \in K$
  - Let $\pi^L \in \Delta(L)$ be its marginal on L
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- For any $p \in \Delta(K)$ let
  - $\Delta_p(\Delta(K))$ be the set of probabilities on $\Delta(K)$ with expectation p
- The set of splittings at π are $A(\pi) := \Delta_p(\Delta(K))$, with $p = \pi^K$ and $B(\pi) := \Delta_q(\Delta(L))$, with $q = \pi^L$

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- The set of splittings at π are $A(\pi) := \Delta_p(\Delta(K))$, with $p = \pi^K$ and $B(\pi) := \Delta_q(\Delta(L))$, with $q = \pi^L$
- $\Phi(\pi, a, b)$ is the unique probability distribution on S induced by π, a and b, which is a splitting of $\Delta_\pi(S)$
- In the independent case, every player controls a separate martingale and $\Phi(\pi, a, b) = a \otimes b$

Notation

For any $f : \Delta(K \times L) \to \mathbb{R}$, $Q \in \Delta(L)^K$ and $P \in \Delta(K)^L$ we set

- $f_K(\cdot, Q) : \Delta(K) \to \mathbb{R}, \quad p \mapsto f(p \otimes Q)$
- $f_L(\cdot, P) : \Delta(L) \to \mathbb{R}, \quad q \mapsto f(q \otimes P)$

f is K-concave if f_K is concave on $\Delta(K)$
f is L-convex if f_L is convex on $\Delta(L)$

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Mertens-Zamir system of equations:

$$f_K(p, Q) = \mathrm{Cav}_{\Delta(K)} \min\{u_K, f_K\}(p, Q), \quad \forall p, Q$$

$$f_L(q, P) = \mathrm{Vex}_{\Delta(L)} \max\{u_L, f_L\}(q, P), \quad \forall q, P$$

The unique solution is denoted $V = MZ(u)$

Main results

Theorem 1. The SG has a value $W_\theta(\pi)$. Moreover

- $\pi \mapsto W_\theta(\pi)$ is K-concave, L-convex and Lipschitz
- $W_\theta(\pi) = \max_{a\in A(\pi)} \min_{b\in B(\pi)} \mathbb{E}[\theta_1 u(\pi') + W_{\theta^+}(\pi')]$

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Elements of the proof

(1) $(\pi, a, b) \mapsto \Phi(\pi, a, b)$ is continuous and bi-linear

(2) Define the dependent splitting operator

$$f \mapsto \varphi(f)(\pi) = \max_{a\in A(\pi)} \min_{b\in B(\pi)} \mathbb{E}_{\Phi(\pi,a,b)}[f(\pi')]$$

(3) Establish a recurrence formula for W_θ^- and W_θ^+

Define the following 4 properties for real functions on $\Delta(K \times L)$

(P1) f is L-convex

(P2) $f_K(p, Q) \leq \mathrm{Cav}_{\Delta(K)} \min\{u_K, f_K\}(p, Q)$ for all p, Q

(Q1) f is K-concave

(Q2) $f_L(q, P) \leq \mathrm{Vex}_{\Delta(L} \max\{u_L, f_L\}(q, P)$ for all q, P

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Theorem 2. Let $f, g : \Delta(K \times L) \to \mathbb{R}$ be such that f satisfies $(P1)$-$(P2)$ and g satisfies $(Q1)$-$(Q2)$. Then

$$f \leq W_\infty^- \leq W_\infty^+ \leq g$$

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Elements of the proof

(1) Let f satisfy $(P1)$-$(P2)$. Define a strategy $\sigma(\varepsilon, f)$, $\pi_m \mapsto a_m$ s.t.

$$\mathbb{E}_{a_m}[\min\{u_K, f_K\}(p, Q_m)] \geq f(\pi_m) - \varepsilon/2^m$$

(2) Define the auxiliary steps $\pi_{m+1/2}$. Work with the martingale $(\pi_{m/2})_{m \geq 1}$

(3) Prove that the strategy $\sigma(\varepsilon, f)$ guarantees $f(\pi) - \varepsilon$ □

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Theorem (MZ 71). The MZ system has a solution v. In particular, $v_K(p, Q) = \mathrm{Cav}_{\Delta(K)} \min\{u_K, v_K\}(p, Q)$ for all p, Q.

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Theorem 3

– The splitting game has a uniform value $W_\infty = v := MZ(u)$

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Theorem (MZ 71). The MZ system has a solution v. In particular, $v_K(p, Q) = \mathrm{Cav}_{\Delta(K)} \min\{u_K, v_K\}(p, Q)$ for all p, Q.

Theorem 3

- The splitting game has a uniform value $W_\infty = v := MZ(u)$
- There exists optimal strategies such that $(\pi_m)_{m\geq 2}$ is constant
- The strategy (for player) 1 is $\sigma(0, v)$ with the additional restriction:
  - (i) If $u(\pi) \geq v(\pi)$, play δ_p
  - (ii) If $u(\pi) < v(\pi)$, play $a = \sum_{r\in R} \lambda^r \delta_{p^r}$ where $\pi^r = p^r \otimes Q$
    $u(\pi^r) = v(\pi^r)$ for all $r \in R$ and $\sum_{r\in R} \lambda^r \min\{u, v\} v(\pi_r) = v(\pi)$

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Elements of the proof

(1) **MZ 71:** The MZ system has a solution v which satisfies $(P1)$-$(P2)$

(2) The strategy $\sigma(0, v)$ guarantees v so that $W_\infty \geq v$

(3) Similarly, one obtains $W_\infty \leq v$

(4) (i) and (ii) ensure that $(\pi_m)_{m\geq 2}$ is constant

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Corollary. $V_\theta \to v = MZ(u)$, as $\sup_m \theta_m \to 0$, where V_θ is the value of the repeated games with incomplete information

Proof: $|W_\theta - V_\theta| \leq C(\sup_m \theta_m)^{1/2}$ for all evaluations θ

Remarks

- In repeated games with incomplete information the uniform value does not exist : each players prefers the other to reveal first
- Although asymptotically equivalent, **a crucial (and surprising) difference is that the Splitting Game has a uniform value**. Observing the other player's use of information makes the game strategically very stable: under optimal play $\Longrightarrow$ at most one splitting

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- The optimal uniform strategy is very simple and "trivializes the game"
- Recently, economists are looking at *commitment strategies* for games with incomplete information, i.e. assume the players can commit to playing some strategy $(\sigma^k)_{k\in K}$. We are then in the splitting game and uniform equilibrium exists

Open problems and possible extensions

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- The splitting game to study non-zero-sum repeated games with incomplete information

Moltes gràcies !

Merci pour votre attention