Coupling of regularization and h-(hp-adaptive) BEM for the delamination problem

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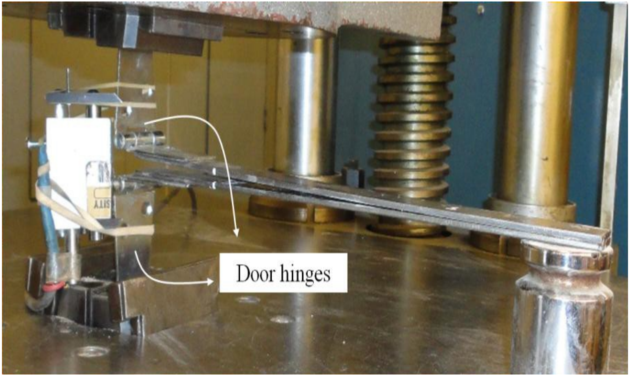
Outline

1. Motivation: Composite structures, Delamination, DCB Test
2. Hemivariational inequalities (HVI), locally Lipschitz functionals
3. Regularization techniques
4. Coupling of regularization and h - (hp-adaptive) BEM
5. Numerical results

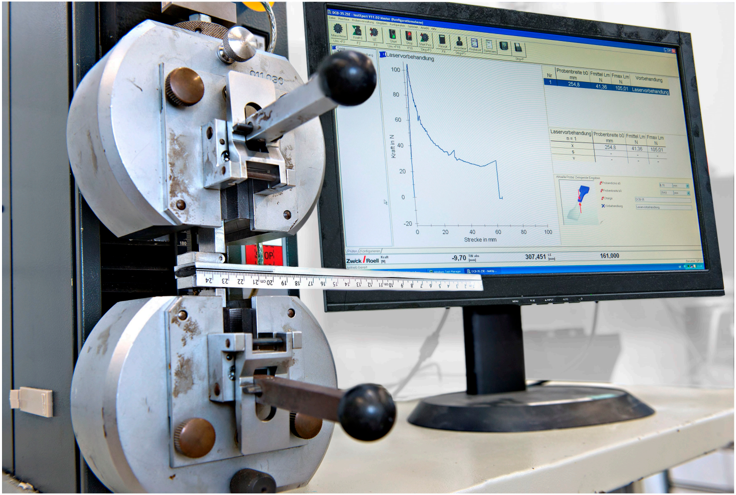
Motivation: Double Cantilever Beam (DCB) Test

Fracture toughness testing

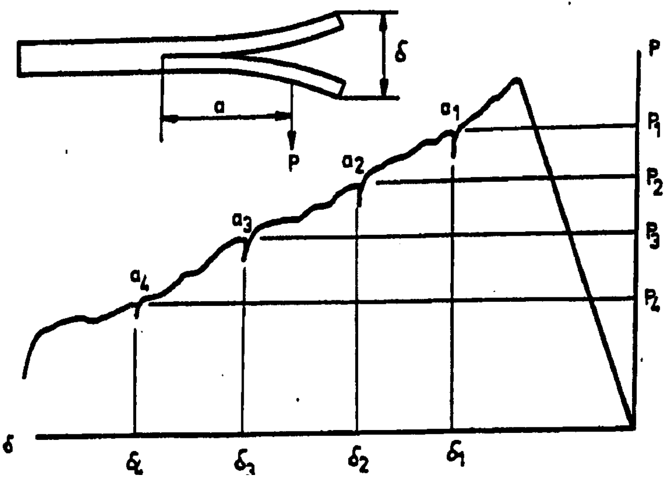
Carbon fiber reinforced polymer (CFRP) project in cooperation with H.-J. Gudladt and his group in material science, UniBw Munich



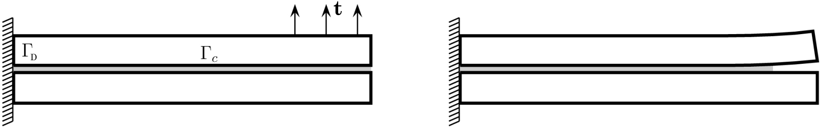
DCB Test Set-up



Experimental Data from DCB Test



Math. Modelling of Delamination Problem



$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \varepsilon(\mathbf{u}) : \sigma(\mathbf{v})\,dx$$

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})),\ \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i})$$

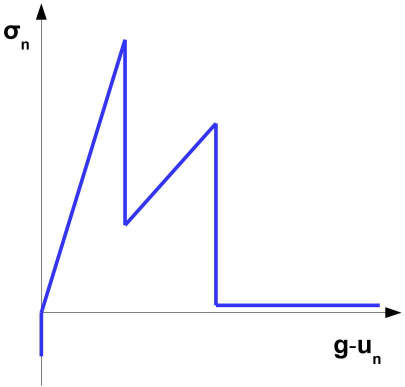
$$\sigma_{ij}(\mathbf{u}) = \frac{E\nu}{1-\nu^2}(\varepsilon_{11}(\mathbf{u}) + \varepsilon_{22}(\mathbf{u}))\,\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}(\mathbf{u})$$

$$\langle \mathbf{L},\mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx + \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{v}\,ds.$$

$$\mathbf{u} = 0 \text{ a.e. on } \Gamma_D$$

Nonmonotone Multivalued Adhesion Law

$-\sigma_n \in \partial j(u_2)$ on Γ_c, j – minimum-type function



$$\sigma_n = \sigma(\mathbf{u})\mathbf{n} \cdot \mathbf{n}, \quad \sigma_t = \sigma(\mathbf{u})\mathbf{n} - \sigma_n \mathbf{n}$$

Delamination Problem

Find $\mathbf{u} \in \mathbf{H}^1(\Omega) := [H^1(\Omega)]^d$, $d = 2, 3$, such that

$$
\begin{aligned}
-\operatorname{div} \sigma(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega \\
\mathbf{u} &= 0 && \text{on } \Gamma_D \\
\sigma(\mathbf{u})\mathbf{n} &= \mathbf{t} && \text{on } \Gamma_N \\
u_n &\leq g && \text{on } \Gamma_C \\
\sigma_t(\mathbf{u}) &= 0 && \text{on } \Gamma_C \\
-\sigma_n(\mathbf{u}) &\in \partial j(u_n) && \text{on } \Gamma_C .
\end{aligned}
$$

$$
\mathcal{K} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \,:\, \mathbf{v}|_{\Gamma_D} = 0,\ v_n \leq g \text{ a.e. on } \Gamma_C\}
$$

Multiplying the equilibrium equation by $\mathbf{v} - \mathbf{u}$, integrating over Ω and applying the divergence theorem yields

$$\int_\Omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v} - \mathbf{u})\, dx = \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx + \int_\Gamma \sigma(\mathbf{u})\mathbf{n} \cdot (\mathbf{v} - \mathbf{u})\, ds.$$

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$$\sigma(\mathbf{u})\mathbf{n}\cdot(\mathbf{v}-\mathbf{u}) = \sigma_t(\mathbf{u})\cdot(\mathbf{v_t}-\mathbf{u_t}) + \sigma_n(\mathbf{u})(v_n-u_n) \quad \text{on } \Gamma_C,$$

$$\sigma_t(\mathbf{u}) = 0, \;\; -\sigma_n(\mathbf{u})(v_n-u_n) \leq j^0(u_n; v_n-u_n) \quad \text{on } \Gamma_C$$

$$\sigma(\mathbf{u})\mathbf{n} = \mathbf{t} \text{ on } \Gamma_N$$

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(Domain HVI): Find $\mathbf{u}\in\mathcal{K}$ such that

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + \int_{\Gamma_C} j^0(u_n(s); v_n(s)-u_n(s))\,ds \ge \langle \mathbf{L}, \mathbf{v}-\mathbf{u}\rangle \;\forall \mathbf{v}\in\mathcal{K}.$$

Uniqueness Assumption

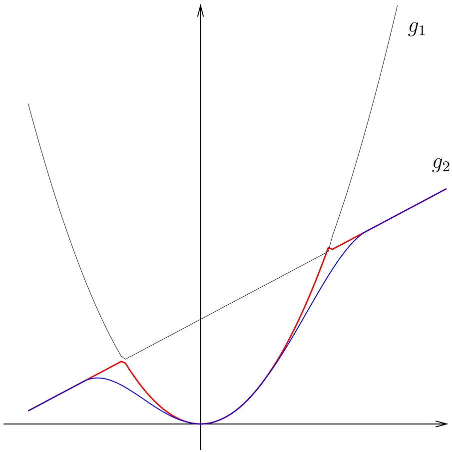
Assume the following one-sided Lipschitz condition on ∂j

$$(\xi^* - \eta^*)\,(\xi - \eta) \geq -\alpha_0 |\xi - \eta|^2 \quad \forall\, \xi^* \in \partial j(\xi),\ \forall\, \eta^* \in \partial j(\eta)$$

for any $\xi, \eta \in \mathbb{R}$ and some $0 \leq \alpha_0 < \mathit{Coercivity}$.

Approach via Regularization

$$j(x) = \min\{g_1(x), g_2(x)\}$$



If $f(x) = \max\{g_1(x), g_2(x)\}$ then $f(x) = g_1(x) + \mathrm{p}[g_2(x) - g_1(x)]$, where $\mathrm{p} : \mathbb{R} \to \mathbb{R}_+$ is the plus function $\mathrm{p}(t) = t^+ = \max\{t, 0\}$.

Using the smoothing approximation via convolution defined by

$$P(t,\varepsilon) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\rho(s)\, ds,$$

the smoothing approximation for the max function takes the form

$$S(x,\varepsilon) := g_1(x) + P\left(g_2(x) - g_1(x), \varepsilon\right).$$

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Since $\min\{g_1(x), g_2(x)\} = -\max\{-g_1(x), -g_2(x)\}$, these type of functions can be handled as above.

Zang Density Function

With

$$\rho(s) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le s \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \text{then}$$

$$P(t,\varepsilon) = \begin{cases} 0 & \text{if} \quad t < -\frac{\varepsilon}{2} \\ \frac{1}{2\varepsilon}(t+\frac{\varepsilon}{2})^2 & \text{if} \quad -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ t & \text{if} \quad t > \frac{\varepsilon}{2}. \end{cases}$$

Regularized Problem

$$j(x) = \min\{g_1(x), g_2(x)\} \sim S(x,\varepsilon)$$

$$J(\mathbf{u}) = \int_{\Gamma_C} j(u_n(s))\, ds \sim J_\varepsilon(\mathbf{u}) = \int_{\Gamma_C} S(u_n(s),\varepsilon)\, ds$$

$$\langle DJ_\varepsilon(\mathbf{u}),\mathbf{v}\rangle = \int_{\Gamma_C} S'_x(u_n(s),\varepsilon)\, v_n(s)\, ds$$

The regularized problem $(\mathcal{P}_\varepsilon)$ now reads: Find $\mathbf{u}_\varepsilon \in K$ such that

$$a(\mathbf{u}_\varepsilon, \mathbf{v}-\mathbf{u}_\varepsilon) + \langle DJ_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{v}-\mathbf{u}_\varepsilon\rangle \geq \langle \mathbf{L}, \mathbf{v}-\mathbf{u}_\varepsilon\rangle \quad \forall \mathbf{v} \in \mathcal{K}.$$

Boundary Integral Operators - In cooperation with L. Banz, University of Salzburg

We introduce

$$(V\phi)(\mathbf{x}) := \int_\Gamma G(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds_y, \qquad (K\phi)(\mathbf{x}) := \int_\Gamma \mathbf{T}_y G^\top(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds_y$$

$$(K'\phi)(\mathbf{x}) := \mathbf{T}_x \int_\Gamma G(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds_y, \quad (W\phi)(\mathbf{x}) := -\mathbf{T}_x\,(K\phi)(\mathbf{x}),$$

where

$$G(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\left(\log|\mathbf{x}-\mathbf{y}|\mathrm{I} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^\top}{|\mathbf{x}-\mathbf{y}|^2}\right), & \text{if d=2} \\ \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\left(|\mathbf{x}-\mathbf{y}|^{-1}\mathrm{I} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^\top}{|\mathbf{x}-\mathbf{y}|^3}\right), & \text{if d=3} \end{cases}$$

and $\mathbf{T}_y(\mathbf{u}) := \sigma(\mathbf{u}(\mathbf{y}))\cdot\mathbf{n}_y$.

Boundary Variational Inequality Formulation via the Poincaré-Steklov operator

With $P := W + \left(K' + \frac{1}{2}\mathrm{I}\right) V^{-1} \left(K + \frac{1}{2}\mathrm{I}\right) \Rightarrow$ (BHVI): Find $\mathbf{u} \in \mathcal{K}^{\Gamma}$ such that for all $\mathbf{v} \in \mathcal{K}^{\Gamma}$,

$$\int_{\Gamma_\Sigma} (P\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})ds + \int_{\Gamma_C} j^0(u_n(s); v_n(s)-u_n(s))ds \geq \langle \mathbf{F}, \mathbf{v}-\mathbf{u}\rangle_{\Gamma_\Sigma},$$

where

$$\langle \mathbf{F}, \mathbf{v}\rangle_{\Gamma_\Sigma} = \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{v}\, ds + \langle N\mathbf{f}, \mathbf{v}\rangle_{\Gamma_\Sigma},$$

$$\Gamma_\Sigma := \Gamma_C \cup \Gamma_N,\ \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma) = \{v = v'|_{\Gamma_\Sigma} : \exists\, v' \in \mathbf{H}^{1/2}(\Gamma),\ \mathrm{supp}\, v' \subset \Gamma_\Sigma\},$$

$$\mathcal{K}^{\Gamma} = \{\mathbf{v} \in \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma) : v_n \leq g \text{ a.e. on } \Gamma_C\}.$$

Uniqueness Result

Theorem 1

Let $\mathbf{u}$ *be a solution of the BHVI problem. Then,* $\mathbf{u}$ *is unique if one of the following conditions holds:*

(a) *The jumps are non-negative and the Lipschitz constant* c_L *(of the superpotential* j*) satisfies* $c_L < c_P$.

(b) *The jump* c_J *is negative,* $c_L < c_P$ *and* $u_n(x) \leq t_J - \frac{1}{\tilde{c}}$ *on* Γ_C *for some positive* $\tilde{c}$ *such that* $\tilde{c} < -\frac{c_P - c_L}{c_J}$.

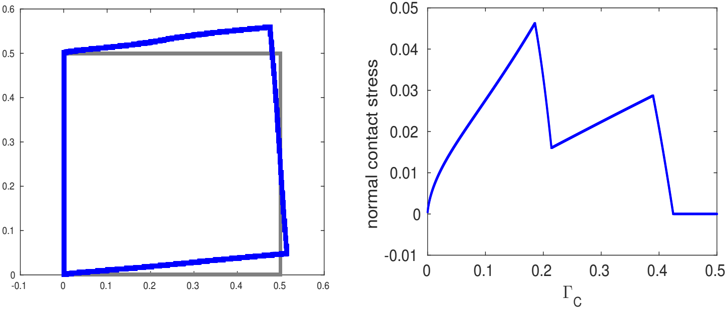
Discretization by Boundary Elements

$$\mathcal{V}_{hp} = \{\mathbf{v_{hp}} \in \mathbf{C}^0(\Gamma) : \mathbf{v_{hp}}|_T \circ \Psi_T \in [\mathbb{P}_{p_T}(\hat{T})]^d \quad \forall T \in \mathcal{T}_h,\ \mathbf{v_{hp}} = 0 \text{ on } \overline{\Gamma}_D\},$$
$$\mathcal{K}^{\Gamma}_{hp} = \{\mathbf{v_{hp}} \in \mathcal{V}_{hp} : (\mathbf{v_{hp}} \cdot \mathbf{n})(P_i) \leq g(P_i) \quad \forall P_i \in \Sigma_{hp}\}$$
$$\mathcal{W}_{hp} = \{\psi_{hp} \in \mathbf{L}^2(\Gamma) : \psi_{hp}|_T \circ \Psi_T \in [\mathbb{P}_{p_T-1}(\hat{T})]^d \quad \forall T \in \mathcal{T}_h\} \subset \mathbf{H}^{-1/2}(\Gamma).$$

The discretized regularized problem $(\mathcal{P}_{\varepsilon,hp})$ is: Find $\mathbf{u}^{\varepsilon}_{hp} \in \mathcal{K}^{\Gamma}_{hp}$ such that for all $\mathbf{v}_{hp} \in \mathcal{K}^{\Gamma}_{hp}$,

$$\langle P_{hp}\mathbf{u}^{\varepsilon}_{hp}, \mathbf{v}_{hp} - \mathbf{u}^{\varepsilon}_{hp}\rangle_{\Gamma_\Sigma} + \langle DJ_\varepsilon(\mathbf{u}^{\varepsilon}_{hp}), \mathbf{v}_{hp} - \mathbf{u}^{\varepsilon}_{hp}\rangle_{\Gamma_C} \geq \langle \mathbf{F}, \mathbf{v}_{hp} - \mathbf{u}^{\varepsilon}_{hp}\rangle_{\Gamma_\Sigma}.$$

Numerical Results



Left: Deformation, *Right:* Normal component of the stress vector along Γ_C

A-priori Error Estimate

Theorem 2

Let $\mathbf{u}_\varepsilon \in \mathcal{K}^\Gamma$, $\mathbf{u}^\varepsilon_{hp} \in \mathcal{K}^\Gamma_{hp}$ *be the solutions of the problems* $(\mathcal{P}_\varepsilon)$ *and* $(\mathcal{P}_{\varepsilon,hp})$, *respectively. Assume that* $\alpha_0 < c_p$, $\mathbf{u}_\varepsilon \in \mathbf{H}^{3/2}(\Gamma)$, $g \in H^{3/2}(\Gamma_C)$ *and* $P\mathbf{u}_\varepsilon - \mathbf{F} \in \mathbf{L}^2(\Gamma)$. *Then there exists a constant* $c = c(\mathbf{u}_\varepsilon, g, \mathbf{f}, \mathbf{t}) > 0$, *but independent of* h *and* p *such that*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}^\varepsilon_{hp}\|_{\mathbf{H}^{1/2}(\Gamma)} \leq c h^{1/4} p^{-1/4}.$$

Mixed Regularized Formulation

Find $(\mathbf{u}^\varepsilon, \lambda^\varepsilon) \in \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma) \times M(\mathbf{u}^\varepsilon)$ such that

$$\langle P\mathbf{u}^\varepsilon, \mathbf{v}\rangle_{\Gamma_\Sigma} + \langle \lambda^\varepsilon, v_n\rangle_{\Gamma_C} = \langle \mathbf{F}, \mathbf{v}\rangle_{\Gamma_\Sigma} \qquad \forall \mathbf{v} \in \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma)$$

$$\langle \mu - \lambda^\varepsilon, u_n^\varepsilon\rangle_{\Gamma_C} \leq 0 \qquad \forall \mu \in M(\mathbf{u}^\varepsilon)$$

with the set of admissible Lagrange multipliers

$$M(\mathbf{u}^\epsilon) := \left\{ \mu \in X^* : \ \langle \mu, \eta\rangle_{\Gamma_C} \geq \langle DJ_\epsilon(\mathbf{u}^\epsilon), \eta\rangle_{\Gamma_C} \ \forall \eta \in X, \ \eta \geq 0 \text{ a.e. on } \Gamma_C \right\},$$

where $X = \{w \mid \exists \mathbf{v} \in \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma), \ v_n|_{\Gamma_C} = w\} \subset H^{1/2}(\Gamma_C)$ and X^* its dual space.

Given the discrete solution $\mathbf{u}_{hp}^{\varepsilon} \in \mathcal{K}_{hp}^{\Gamma}$ to $(\mathcal{P}_{\varepsilon,hp})$, we reconstruct $\lambda_{hp}^{\varepsilon}$ such that

$$\left\langle \lambda_{hp}^{\varepsilon}, v_n \right\rangle_{\Gamma_C} = \langle \mathbf{F}, \mathbf{v} \rangle_{\Gamma_\Sigma} - \left\langle P_{hp} \mathbf{u}_{hp}^{\varepsilon}, \mathbf{v} \right\rangle_{\Gamma_\Sigma} \quad \forall\, v \in \mathcal{V}_{hp}$$

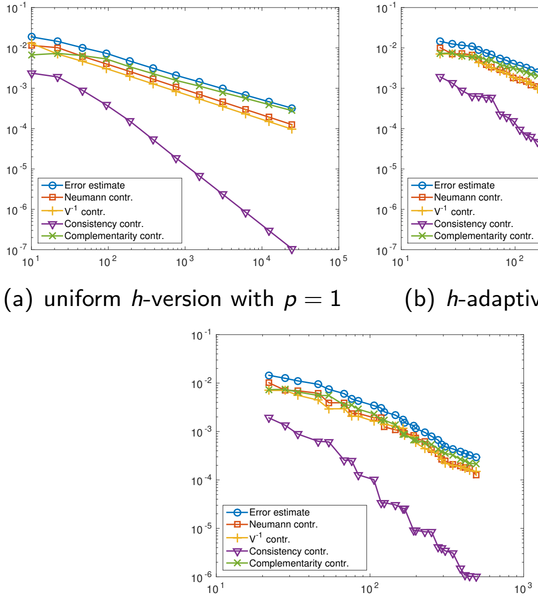
Auxiliary problem: Find $\mathbf{z} \in \mathcal{V} := \tilde{\mathbf{H}}^{1/2}(\Gamma_\Sigma)$ such that

$$\langle P\mathbf{z}, \mathbf{v} \rangle_{\Gamma_\Sigma} = \langle \mathbf{F}, \mathbf{v} \rangle_{\Gamma_\Sigma} - \left\langle \lambda_{hp}^{\varepsilon}, v_n \right\rangle_{\Gamma_C} \quad \forall\, \mathbf{v} \in \mathcal{V}$$

Theorem 1 (A-posteriori error estimate)

If $S_x(\cdot, \varepsilon)$ is Lipschitz continuous, then there exists a constant C independent of h and p such that for $\varsigma > 0$ arbitrary

$$\begin{aligned}(c_P - \alpha_0 - 4\varsigma) \left\| \mathbf{u}^{\varepsilon} - \mathbf{u}_{hp}^{\varepsilon} \right\|_{\mathcal{V}}^2 \leq \left(\frac{C}{\varsigma} + 1 \right) \left\| \mathbf{z} - \mathbf{u}_{hp}^{\varepsilon} \right\|_{\mathcal{V}}^2 &+ \frac{1}{4\varsigma} \left\| (\lambda_{hp}^{\varepsilon} - DJ_{\varepsilon}(\mathbf{u}_{hp}^{\varepsilon}))^- \right\|_{X^*}^2 \\ &+ C \left(\frac{1}{\varsigma} + \frac{1}{\beta^2} + \frac{1}{\varsigma \beta^2} \right) \left\| (u_{hp,n}^{\varepsilon} - g)^+ \right\|_X^2 \\ &- \left\langle (\lambda_{hp}^{\varepsilon} - DJ_{\varepsilon}(\mathbf{u}_{hp}^{\varepsilon}))^+, (u_{hp,n}^{\varepsilon} - g)^- \right\rangle_{\Gamma_C}.\end{aligned}$$



(a) uniform h-version with $p = 1$

(b) h-adaptive with $p = 1$

(c) hp-adaptive

- N. Ovcharova: On the coupling of regularization techniques and the boundary element method for a hemivariational inequality modelling a delamination problem, Math. Methods Appl. Sci., arXiv 1603.05091 (2015)
- N. Ovcharova, L. Banz: Coupling regularization and adaptive *hp*-BEM for the solution of a delamination problem, Numerische Mathematik, arXiv 1510.06343 (2015)

Thank you very much for your attention!