Smoothness of the Metric Projection onto Nonconvex Bodies in Hilbert Spaces

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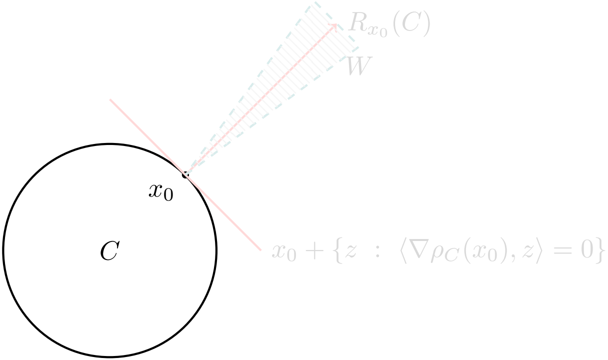
Contents

Theorem 1.1 (Holmes, 1973)

Let C be a convex body of a Hilbert space X such that $0 \in \operatorname{int} C$ and $\operatorname{bd} C$ is a $\mathcal{C}^{p+1}$-submanifold near $x_0 \in \operatorname{bd} C$. Let ρ_C be the Minkowski functional of C. Then, there exists an open neighborhood W of the open normal ray

$$R_{x_0}(C) = \{x_0 + t\nabla\rho_C(x_0) \; : \; t > 0\},$$

such that $P_C(\cdot)$ is of class $\mathcal{C}^p$ on W.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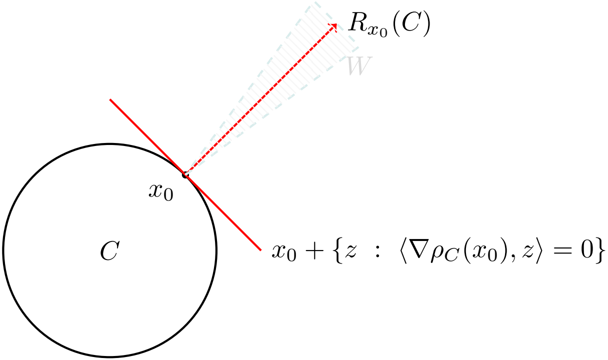
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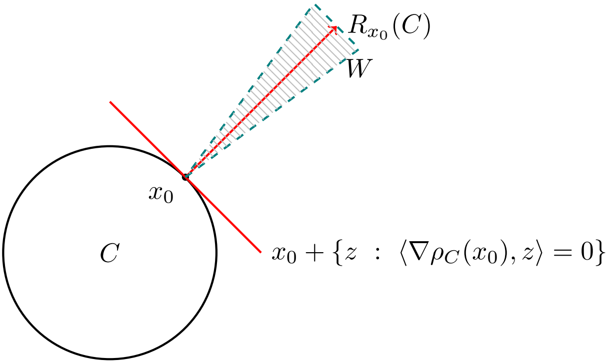

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Elements of the Proof:

(i) The smoothness of $\operatorname{bd} C$ at x_0 is equivalent to the smoothness of $\rho_C(\cdot)$ at x_0 (regardless possible translations).

(ii) Furthermore, the exterior normal vector of $\operatorname{bd} C$ at x_0 is $\nu = \nabla\rho_C(x_0)/\|\nabla\rho_C(x_0)\|$.

(ii) The distance function $d_C(\cdot)$ is of class $\mathcal{C}^1$ on $X \setminus C$.

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(ii) The distance function $d_C(\cdot)$ is of class $\mathcal{C}^1$ on $X \setminus C$.

(iv) At each point $u_0 \in R_{x_0}(C)$, we can choose two neighborhoods $U \in \mathcal{N}(u_0)$ and $V \in \mathcal{N}(x_0)$ such that the function

$$F : U \times V \to X$$
$$(u, v) \mapsto u - v - d_C(u)\frac{\nabla \rho_C(v)}{\|\nabla \rho_C(v)\|}$$

satisfies that $F(u, v) = 0$ if and only if $v = P_C(u)$.

(v) $D_2F(u_0, x_0)$ is invertible, and therefore, we can apply the Implicit Function Theorem.

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Definition 2.1

For a closed set S, a point $x_0 \in S$ and a point $u \in X$ we define:

(a) $\mathrm{Proj}_S(u) = \{s \in S \ : \ d_S(u) = \|u - s\|\}$. *When* $\mathrm{Proj}_S(u)$ *is a singleton, we write* $P_S(u)$ *instead.*

(b) *The* **Proximal normal cone** *of* S *at* x_0 *as*

$$N^P(S; x_0) = \{\zeta \in X \ : \ \exists t > 0, \ x_0 \in \mathrm{Proj}_S(x_0 + t\zeta)\}.$$

(c) Whenever the Proximal normal cone of S at x_0 has the form

$$N^P(S;x_0) = \{t\nu \ : \ t \geq 0\},$$

for some $\nu \in \mathbb{S}_X$, we define the **open normal ray** and the λ-**truncated open normal ray** of S at x_0 as

$$R_{x_0}(S) = \{x_0 + t\nu \ : \ t > 0\}$$
$$R_{x_0,\lambda}(S) = \{x_0 + t\nu \ : \ t \in (0,\lambda)\}.$$

Definition 2.2 (Prox-Regular sets)

For $r \in (0, +\infty]$ *and* $\alpha > 0$*, we say that* S *is* **(r, α)-prox-regular at x_0** *if for every* $x \in S \cap B_X(x_0, \alpha)$ *and every* $\zeta \in N^P(S; x) \cap \mathbb{B}_X$ *we have that*

$$x \in \mathrm{Proj}_S(x + t\zeta), \quad \textit{for every real } t \in [0, r]. \tag{1}$$

We say that S *is* **r-prox-regular at x_0** *if it is* (r, α)*-prox-regular at* x_0 *for some* $\alpha > 0$*.*

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Definition 2.3

For $r \in (0, +\infty]$ and $\alpha > 0$ we define the sets

$$\mathcal{R}_S(x_0, r, \alpha) := \left\{ x + tv \ : \ \substack{x \in S \cap B_X(x_0, \alpha), \\ t \in [0, r), \ v \in N^P(S; x) \cap \mathbb{B}_X} \right\},$$

$$\mathcal{W}_S(x_0, r, \alpha) := \left\{ u \in X \ : \ \substack{\mathrm{Proj}_S(u) \cap B_X(x_0, \alpha) \neq \emptyset, \\ d_S(u) < r} \right\}.$$

In general, $\mathcal{R}_S(x_0, r, \alpha) \supset \mathcal{W}_S(x_0, r, \alpha)$, but the equality doesn't always hold.

Theorem 2.4 (Mazade, 2011)

The following assertions are equivalent:

(i) S *is* (r, α)*-prox-regular at* x_0*;*

(ii) $\mathcal{W}_S(x_0, r, \alpha)$ *is open and* d_S *is* $\mathcal{C}^1$ *on* $\mathcal{W}_S(x_0, r, \alpha) \setminus S$ *with*

$$\nabla d_S(u) = \frac{u - P_S(u)}{d_S(u)};$$

(iii) *For any* $x \in S \cap B(x_0, \alpha)$ *and* $\zeta \in N^P(S; x)$ *one has*

$$\langle \zeta, x' - x \rangle \leq \frac{\|\zeta\|}{2r} \|x' - x\|^2 \quad \text{for all } x' \in S.$$

Moreover, if S *is* (r, α)*-prox-regular at* x_0*, then* $\mathcal{R}_S(x_0, r, \alpha)$ *and* $\mathcal{W}_S(x_0, r, \alpha)$ *coincide.*

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Theorem 3.1

Let $O_0 \subseteq X$ be an open set and $f : O_0 \subseteq X \to \mathbb{R}$ be a function of class $\mathcal{C}^{p+1}$ near $x_0 \in X$ such that $\nabla f(x_0) = 0$. Assume that $\overline{\operatorname{epi} f}$ is r-prox-regular at $(x_0, f(x_0))$. For the constant

$$\lambda = \min \left\{ r, \ \left(-2 \inf \left\{ \langle u, D^2 f(x_0) u \rangle \ : \ u \in \mathbb{B}_X \right\} \right)^{-1} \right\}$$

there exists an open neighborhood W of $R_{(x_0, f(x_0)), \lambda}(\operatorname{epi} f)$ such that

(a) *$d_{\operatorname{epi} f}$ is of class $\mathcal{C}^{p+1}$ on W;*

(b) *$P_{\operatorname{epi} f}$ is of class $\mathcal{C}^{p}$ on W.*

Sketch of proof

Denote $S := \overline{\operatorname{epi} f}$ and $v_0 := (x_0, f(x_0))$. We will write $u = (u_1, u_2)$ for every $u \in X \times \mathbb{R}$.

For $\alpha > 0$ small enough and $O := \mathcal{W}_S(v_0, r, \alpha) \subset X \times \mathbb{R}$ we can ensure:

1. $\pi_X(O) \subseteq O_0$, f is $\mathcal{C}^{p+1}$ on $\pi_X(O)$.
2. $d_S(\cdot)$ is $\mathcal{C}^1$ on O.
3. For $x \in \pi_X(O)$, $N^P(S; (x, f(x))) = \{t(\nabla f(x), -1) \ : \ t \geq 0\}$.
4. $P_S\left[\left(v + N^P(S; v)\right) \cap O\right] = v$, for $v \in S \cap O$.
5. $R_{v_0,\lambda}(S) \subset O$.

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Fix $u_0 \in R_{v_0,\lambda}(S)$ and choose $U \in \mathcal{N}_X(u_0)$ and $V \in \mathcal{N}(v_0)$ with $U, V \subseteq O$. Define

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\begin{aligned}
F : U \times V &\to X \times \mathbb{R} \\
(u, v) &\mapsto u - v - d_S(u)\varphi(v),
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where $\varphi(v) = \frac{(\nabla f(v_1), -1)}{\|(\nabla f(v_1), -1)\|}$ for all $v \in V$.

If we choose correctly U and V, we can assure that

$$F(u, v) = 0 \iff v = P_S(u).$$

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If we choose correctly U and V, we can assure that

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It is not hard to see that

$$D_2F(u_0, v_0) = -(\mathrm{id}_{X\times\mathbb{R}} + d_S(u)D\varphi(v_0)),$$

and, using that $\nabla f(x_0) = 0$, we can show by simple computation that

$$D\varphi(v_0)h = (D^2 f(x_0)h_1, 0),\ \forall h \in X \times \mathbb{R}.$$

Using that $d_S(u_0) < \lambda$, we prove that $D_2F(u_0, v_0)$ is bijective, and so we can apply the **Implicit Function Theorem (IFT)**:

IFT: d_S is of class $\mathcal{C}^1$ on U, and so F is of class $\mathcal{C}^1$ on $U \times V$. Then, there exist $U_1 \in \mathcal{N}(u_0)$ and $V_1 \in \mathcal{N}(v_0)$ and a mapping $\phi : U_1 \to V_1$ such that

(i) ϕ is of class $\mathcal{C}^1$;

(ii) For each $u' \in U_1$, $F(u', \phi(u')) = 0$;

(iii) For each $(u', v') \in U_1 \times V_1$, $F(u', v') = 0 \implies v = \phi(u')$.

We can see that $\phi = P_S$ on U_1 and so P_S is of class $\mathcal{C}^1$ and d_S is of class $\mathcal{C}^2$.

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We can apply recursively this argument as follows:

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\begin{aligned}
d_S \text{ is } \mathcal{C}^2 \text{ in } U_1 &\implies F \text{ is } \mathcal{C}^2 \text{ on } U_1 \times V_1 \\
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We cannot go further, since $\varphi(\cdot)$ is only $\mathcal{C}^p$, and so is F.

The proof is finished considering W as the union of the neighborhoods U_p obtained for each $u_0 \in R_{v_0,\lambda}(S)$.

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Theorem 3.2

Let $S \subseteq X$ be a closed body near $x_0 \in \operatorname{bd} S$. Assume that there exist $r \in (0, +\infty]$ and $\alpha > 0$ such that $B_X(x_0, \alpha) \cap \operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold and that S is r-prox-regular at x_0.

Then there exists a neighborhood V of $R_{x_0,r}(S)$ such that

- *d_S is of class $\mathcal{C}^{p+1}$ on V;*
- *P_S is of class $\mathcal{C}^p$ on V.*

Furthermore, if the set S is (α, r)-prox-regular at x_0, then

- *d_S is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$;*
- *P_S is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$.*

Sketch of proof:

We will prove only the second part, which entails the first one.

- We can regard (near x_0) the set S as the epigraph of a $\mathcal{C}^{p+1}$ function $f : O_0 \subseteq Z \to \mathbb{R}$, where $Z := T_{x_0}(\operatorname{bd} S)$ (which is an hyperplane). In particular, we have that $\nabla f(z_0) = 0$, where $z_0 := \pi_Z(x_0)$.
- Considering $X = Z \times \mathbb{R}$ and noting that $x_0 = (z_0, f(z_0))$, we can choose O_0 small enough such that $\overline{\operatorname{epi} f}$ is r-prox-regular at $(z_0, f(z_0))$.

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- We can prove, thanks to the r-prox-regularity, that

$$\inf \left\{ \langle z, D^2 f(z_0) z \rangle \ : \ z \in \mathbb{B}_Z \right\} \geq -\frac{1}{r},$$

  and so, the constant λ in Theorem 3.1 is greater than $r/2$.

- Using Theorem 3.1, d_S is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0, r/2, \alpha) \setminus S$.

Now, take $x \in B_X(x_0, \alpha) \cap \operatorname{bd} S$ and $u \in R_{x,r}(S)$ with $d_S(u) \geq \frac{r}{2}$. We can choose $\alpha' > 0$ small enough such that $B_X(x, \alpha')$ is included in $B_X(x_0, \alpha)$ and S is (r, α')-prox-regular at x.

We get that $d_S(\cdot)$ is of class $\mathcal{C}^{p+1}$ in $\mathcal{W}_S(S, r/2, \alpha') \setminus S$ and so:

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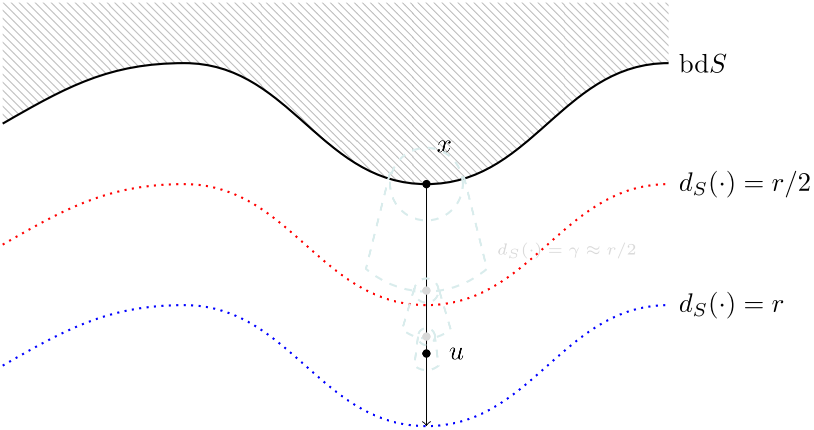
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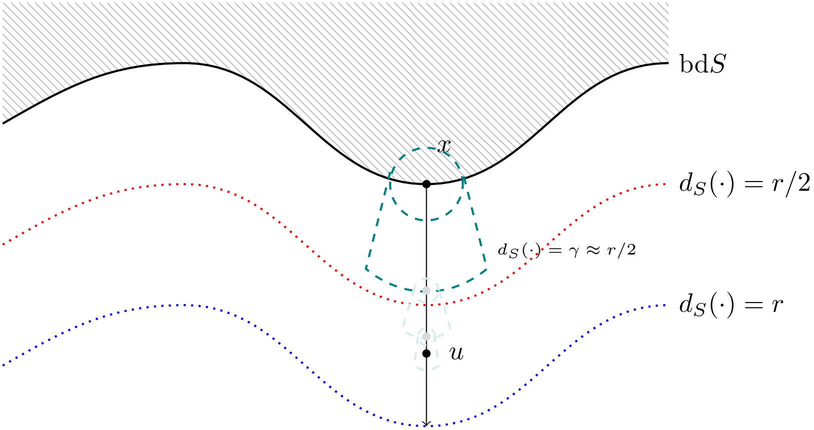
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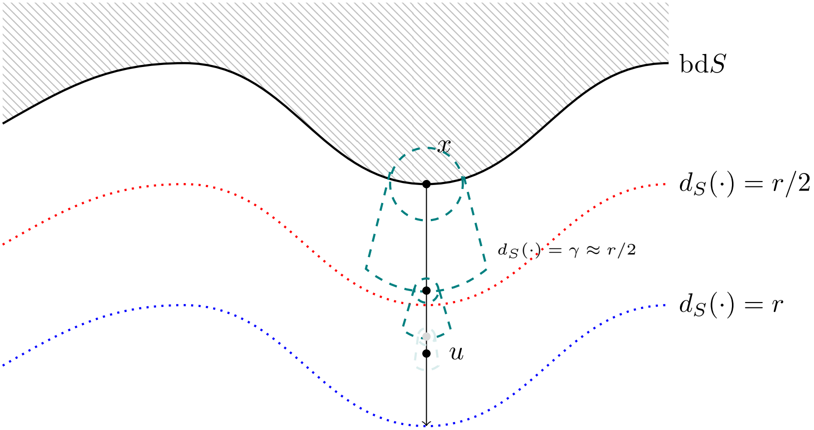
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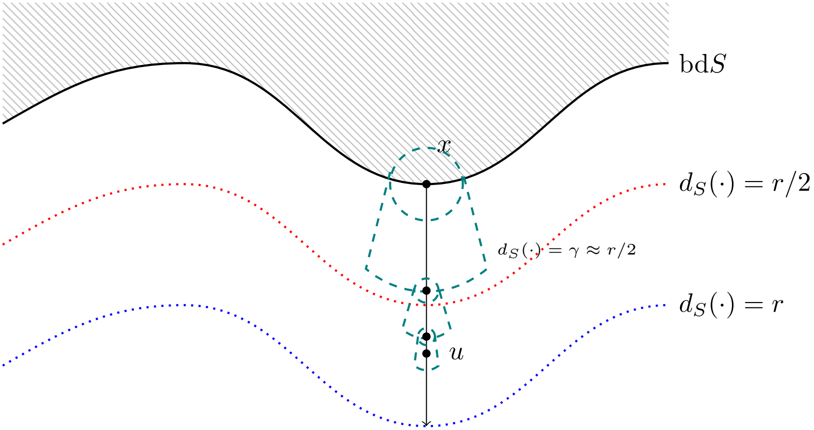
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Since u is arbitrary, we conclude that $d_S(\cdot)$ is of class $\mathcal{C}^{p+1}$ at each element of the set

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- With L. Thibault, we proved the converse of Theorem 3.2, adding an extra hypothesis. We followed the ideas of Fitzpatrick and Phelps [2]. This is an ongoing work.
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Some comments:

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Thank you for your attention.