Chance constrained optimization of a space launcher

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Journées SMAI-MODE - Toulouse, 25/03/2016

Introduction

Problem

Deliver a payload to a given altitude while **minimizing** the fuel load of the launcher.

Some parameters are subject to **uncertainties** and we need the mission to succeed with a **90% probability**.

Framework

General formulation

$$\begin{cases} \text{Compute} \\ \quad \min_{x \in X} J(x) \\ \text{Subject to} \\ \quad \mathbb{P}\big[G(x,\omega) \geq 0\big] \geq p \end{cases}$$

$x \in X \subseteq \mathbb{R}^n$ (optimization variables)

$\omega \in \Omega \subseteq \mathbb{R}^m$ (random parameters)

$p \in [0,1]$ (probability threshold)

$J : \mathbb{R}^n \to \mathbb{R}$ (cost)

$G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ (constraint)

Main theoretic result

Definition: quasi-concave function

A function $f(z)$, $z \in \mathbb{R}^n$ is said to be quasi-concave if, for any $z_1, z_2 \in \mathbb{R}^n$ and $\lambda \in (0,1)$, the following inequality holds

$$f\left(\lambda z_1 + (1-\lambda) z_2\right) \geq \min\left\{f(z_1), f(z_2)\right\}$$

Theorem (A. Prékopa, 1995)

If $G(x,y)$ is a quasi-concave function of the variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\omega \in \mathbb{R}^m$ is a random variable with logconcave probability distribution, then the function

$$P(x) := \mathbb{P}\left[G(x,\omega) \geq 0\right] \quad x \in \mathbb{R}^n$$

is logconcave.

Approach

Problem:

for every x in X, the **distribution** of $G(x, \omega)$ is **unknown**.

Solution:

approximate it and translate the stochastic optimization problem into a deterministic one:

$$1 - \mathbb{P}[G(x,\omega) \geq 0] = \int_{-\infty}^{0} f_{G(x)}(\xi) d\xi \approx \int_{-\infty}^{0} \hat{f}_{G(x)}(\xi) d\xi$$

where f_G is the **probability density function** of G and $\hat{f}_G$ its approximation.

Approach

Kernel Density Estimation

Let $\{s_1, s_2, \ldots, s_n\}$ be a sample of size m from the random variable s. A **Kernel Density Estimator** for f is the function

$$\hat{f}(\sigma) := \frac{1}{mh} \sum_{i=1}^{m} K\left(\frac{\sigma - s_i}{h}\right)$$

$$K : \mathbb{R} \to \mathbb{R} \qquad \text{(kernel)}$$
$$h > 0 \qquad \text{(bandwidth)}$$

There isn't an explicit formula for the **error** between f and $\hat{f}$[1].

[1] S. J. Sheather. "Density Estimation". In: *Statistical Science* 19(4) (2004), pp. 588–597.

Model

Vertical ascent of a single stage launcher

State equation:

$$\begin{cases} \dot{r}(t,u) = v(t,u) & \text{(altitude)} \\ \dot{v}(t,u) = \frac{T}{m(t,u)}u(t) - g & \text{(speed)} \\ \dot{m}(t,u) = -\frac{T}{v_e}u(t) & \text{(mass)} \end{cases}$$

- g is the gravitational acceleration;
- T is the engine thrust;
- v_e is the fuel speed.

Control:

$$u \in \mathcal{U} := \{u : [0,+\infty) \to [0,1] \subset \mathbb{R} \mid u \text{ is measurable}\}$$

Vertical ascent of a single stage launcher

Initial conditions:

$$\begin{cases} r(0,u) = 0 & \text{(altitude)} \\ v(0,u) = 0 & \text{(speed)} \\ m(0,u) = (1+k)m_{\mathrm{e}} + m_{\mathrm{p}} & \text{(mass)} \end{cases}$$

- k is the stage index;
- m_{e} is the fuel mass;
- m_{p} is the payload.

Deterministic optimization problem

Problem 1

$$\begin{cases} \text{Compute} \\ \quad \max_{u \in \mathcal{U}} m(t_\mathrm{f}, u) \\ \text{Subject to} \\ \quad r(t_\mathrm{f}, u) \geq r_\mathrm{f} \end{cases}$$

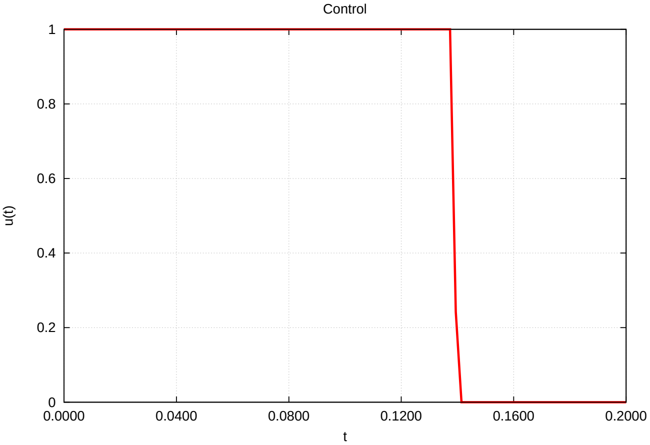
- t_f is the fixed final time;
- r_f is the target final altitude.

Solution

Optimal cost:

$$m(t_\mathsf{f}, u^*) \approx 4.614$$

Optimal control:



Stochastic optimization problem

Stochastic optimization problem

Problem 2a

$$\begin{cases} \text{Compute} \\ \quad \max_{u \in \mathcal{U}} \mathbb{E}\left[m(t_\mathrm{f}, u)\right] \\ \text{Subject to} \\ \quad \mathbb{P}\big[R_\mathrm{f}(T, u) \geq r_\mathrm{f}\big] \geq p \\ \quad T \sim U\Big(\big[\overline{T}(1 - \Delta T), \overline{T}(1 + \Delta T)\big]\Big) \end{cases}$$

- p is the probability threshold.

Constraint:
For a given realization of T

$$R_\mathrm{f}(T, u) := r(t_\mathrm{f}, u)$$

Reformulation

For every $u \in \mathcal{U}$ we have

$$\mathbb{P}\left[R_{\mathrm{f}}(T, u) \geq r_{\mathrm{f}}\right] = 1 - \int_0^{r_{\mathrm{f}}} f_u(\sigma) d\sigma =: 1 - F_u(r_{\mathrm{f}})$$

$$\mathbb{E}\left[m(t_{\mathrm{f}}, u)\right] = \int_0^{t_{\mathrm{f}}} m(0, u) - \frac{\mathbb{E}[T]}{v_{\mathrm{e}}} u(t) dt =: \overline{m}(t_{\mathrm{f}}, u)$$

$F_u(r_{\mathrm{f}})$ is the **probability distribution function** of R_{f}, parameterized by u.

Reformulation

Approximation of f_u:

- choose $n \in \mathbb{N}$ draw a sample from T

$$\{T_1, T_2, \ldots, T_n\}$$

- choose a **kernel** K, a **bandwidth** h and define the **Kernel Density Estimator** of $f_{\mathbf{m}_e}$ as

$$\hat{f}_u(\sigma) := \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{\sigma - R_f(T_i, u)}{h}\right)$$

Deterministic optimization problem

Problem 2b

$$\begin{cases} \text{Compute} \\ \quad \max_{u \in \mathcal{U}} \overline{m}(t_{\mathrm{f}}, u) \\ \text{Subject to} \\ \quad \hat{F}_u(r_{\mathrm{f}}) \leq 1 - p \end{cases}$$

$$\hat{F}_u(r_{\mathrm{f}}) := \int_0^{r_{\mathrm{f}}} \hat{f}_u(\sigma) d\sigma \approx \int_0^{r_{\mathrm{f}}} f_u(\sigma) d\sigma =: F_u(r_{\mathrm{f}})$$

Results

Choice of parameters

Parameter	Value
$\overline{T}$	150
ΔT	0.1
g	9.8
m_{p}	0.5
r_{f}	0.2
p	0.9

kernel:

$$K(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

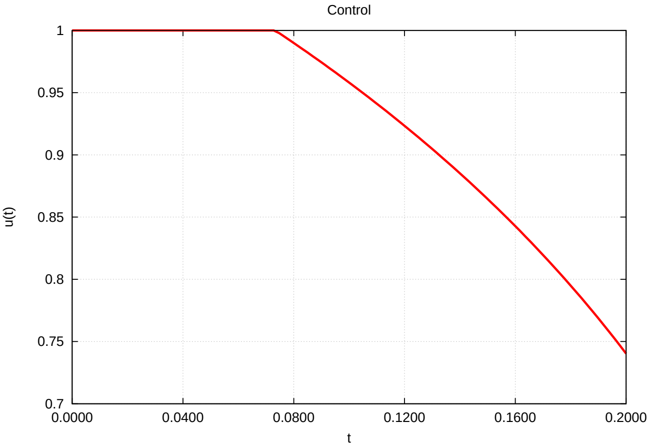
bandwidth:

$$h = 1.06 n^{-\frac{1}{5}} \sigma_n$$

- σ_n is the sample standard deviation.

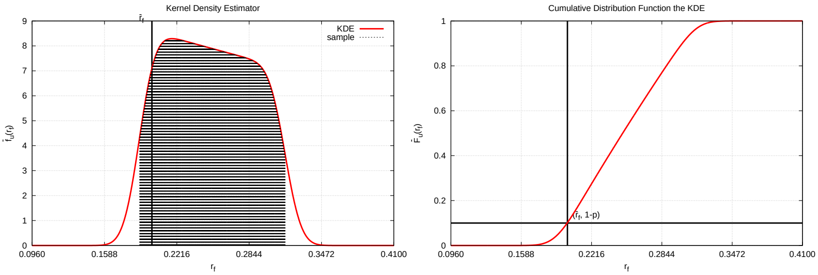
Optimal solution

For a **uniform** sample from T of size $n = 500$ we obtain



allowing us to deliver the payload with a **probability of 90.887%** even if the engine thrust T is subject to **random oscillations**.

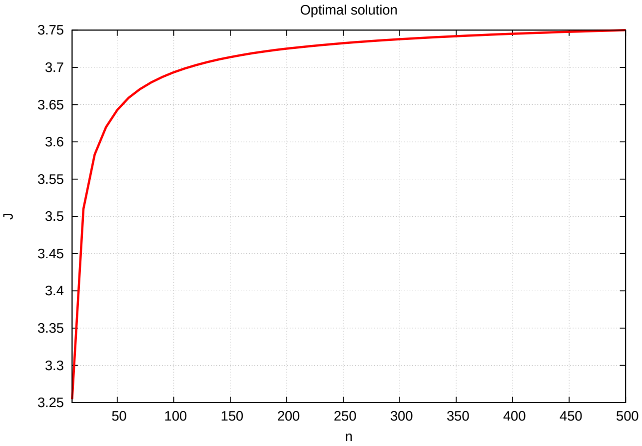
Optimal solution



Approximations of **density** and **distribution** functions for $n = 500$.

Convergence of approximated solutions

The problem is solved for all $n \in \{10, 20, \ldots, 500\}$.



Optimal cost as a function of n.

Convergence of approximated solutions

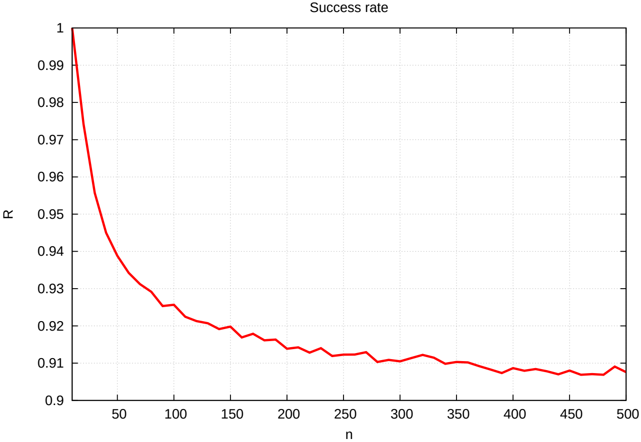
Let u^n be the optimal control obtained with a sample of size n. In order to **estimate** $\mathbb{P}\big[R_{\mathrm{f}}(T, u^n) \geq r_{\mathrm{f}}\big]$ we evaluate $R_{\mathrm{f}}(T, u^n)$ at 10^5 random values of T, then define the **success rate**

$$R_n := \frac{\#\{T_i \quad \text{s.t.} \quad R_{\mathrm{f}}(T_i, u^n) \geq r_{\mathrm{f}}\}}{10^5}$$

and use the fact that

$$R_n \approx \mathbb{P}\big[R_{\mathrm{f}}(T, u^n) \geq r_{\mathrm{f}}\big]$$

Convergence of approximated solutions

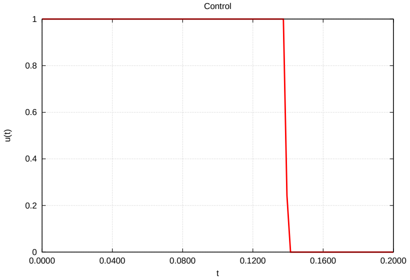


Success rate as a function of n.

Conclusions

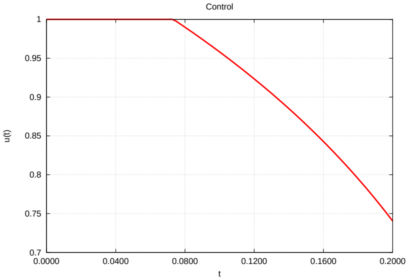
Conclusions

Deterministic problem:



Bang-bang control.

Stochastic problem:



Continuous control.

Conclusions

Pros:

- **Efficiency**: small samples lead to good approximations of f. Better results can be obtained with different h and K.

Cons:

- **Lack of theory**: no explicit formula for the error between f and $\hat{f}$. No general criterion for choosing h and K.

Future work:

- **More random variables**: use realistic models with an increasing number of uncertain parameters.

Fin