Linearization methods for discontinuous control problems

Oana Silvia Serea

University of Perpignan, France

oana-silvia.serea@univ-perp.fr

http://perso.univ-perp.fr/oserea/

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Outline

- We consider the dynamics

$$\begin{cases} dX_s^{t,x,u} = f\left(X_s^{t,x,u}, u_s\right) dt, \ t \leq s \leq T, \\ X_s^{t,x,u} = x \in \mathbb{R}^N, \end{cases}$$

- U compact metric space
- admissible control $u \in \mathcal{U}$: Lebesque-measurable, U-valued
- $f : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$
- (i) f bounded, uniformly continuous
- (ii) $\exists c > 0$ s.t. $|f(x,u) - f(y,u)| \leq c|x - y|, \ \forall (x,y,u) \in \times\mathbb{R}^{2N} \times U$

- Consider $t > 0$ and $x \in \mathbb{R}^N$
- To every $r > t$ and $u \in \mathcal{U}$, we associate

$$\gamma^{t,r,x,u} = \left(\gamma_1^{t,r,x,u}, \gamma_2^{t,r,x,u}\right) \in \mathcal{P}\left([t,r] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$$

defined by

$$\begin{cases} \gamma_1^{t,r,x,u}(A \times B \times C) = \frac{1}{r-t} \int_t^r 1_{A\times B\times C}\left(s, X_s^{t,x,u}, u_s\right) ds, \\ \gamma_2^{t,r,x,u} = \delta_{X_r^{t,x,u}} \end{cases}$$

for all Borel sets $A \subset [t,r]$, $B \subset \mathbb{R}^N$ and $C \subset U$. Here, $\delta(\cdot)$ stands for the Dirac measure

- One can also define $\left(\gamma_1^{t,t,x,u}, \gamma_2^{t,t,x,u}\right) \in \mathcal{P}\left(\{t\} \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$ by

$$\gamma_1^{t,t,x,u} = \delta_{t,x,u_t}, \; \gamma_2^{t,t,x,u} = \delta_x$$

- For every $r \geq t$, we denote by

$$\Gamma(t,r,x) = \left\{ \left(\gamma_1^{t,r,x,u}, \gamma_2^{t,r,x,u}\right) \text{ for all } u \in \mathcal{U} \right\}$$

  and by

- $\Theta(t,r,x) =$

$$\left\{ \begin{array}{l} (\gamma_1,\gamma_2) \in \mathcal{P}\left([t,r]\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^N\right),\ \forall\phi\in C_2^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N\right) \\ \int\limits_{[t,r]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left(\phi(t,x)+(r-t)\mathcal{L}^u\phi(s,y)-\phi(r,z)\right)\gamma_1(dsdydu)\gamma_2(dz)=0 \\ \int\limits_{\mathbb{R}^N} |y|^{2+\delta}\gamma_1([t,r],dy,U) \leq c_{T,x},\ \int\limits_{\mathbb{R}^N} |z|^{2+\delta}\gamma_2(dz) \leq c_{T,x} \end{array} \right\}$$

- The set $C_2^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N\right)$ stands for the set of continuously differentiable functions such that the function and its first order derivatives have at most quadratic growth and

$$\mathcal{L}^u\phi(s,y) = \langle f(y,u), D\phi(s,y)\rangle + \partial_t\phi(s,y)$$

  for all $s \geq 0, y \in \mathbb{R}^N$.

- Note that $\Gamma(t,r,x) \subset \Theta(t,r,x)$

$$-\phi(t,x) + \int_{\mathbb{R}^N} \phi(r,z)\,\gamma_2^{t,r,x,u}(dz) = -\phi(t,x) + \phi\left(r, X_r^{t,x,u}\right) =$$

$$\int_t^r \left(\partial_t \phi\left(s, X_s^{t,x,u}\right) + \left\langle f\left(X_s^{t,x,u}, u_s\right), D\phi\left(s, X_s^{t,x,u}\right)\right\rangle\right) ds =$$

$$\int_{[t,r]\times\mathbb{R}^N\times U} (r-t)\,\mathcal{L}^u\phi(s,y)\,\gamma_1^{t,r,x,u}(dsdydu)$$

  for regular test functions $\phi \in C^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$

- For the deterministic and in the stochastic setting see: Fleming & Vermes '89, Bhatt & Borkar '96, Gaitsgory & Leizarowitz '99, Gaitsgory & Nguyen '02, Gaitsgory '04, Gaitsgory & Rossomakhine '06, Borkar & Gaitsgory '07, Finlay & al. '07, Gaitsgory & Quincampoix '09

- Consider $l : \mathbb{R}^N \times U \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^N \longrightarrow \mathbb{R}$
- One associates the value function

$$V_{l,g}^r(t,x) = \inf_{u \in \mathcal{U}} \left(\int_t^r l\left(x_s^{t,x,u}, u_s\right) ds + g\left(x_r^{t,x,u}\right) \right)$$

- the linearized problem

$\Lambda_{l,g}^r(t,x) =$

$$\inf_{\gamma=(\gamma_1,\gamma_2)\in\Theta(t,r,x)} \left((r-t) \int_{[t,r]\times\mathbb{R}^N\times U} l(y,u)\,\gamma_1(dsdydu) + \int_{\mathbb{R}^N} g(z)\,\gamma_2(dz) \right)$$

- and its dual

$$\eta_{l,g}^{r}(t,x) =$$

$$\sup \left\{ \begin{array}{l} \eta \in \mathbb{R} : \exists \phi \in C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right) \text{ s.t.} \\ \eta \leq (r-t)\left[\mathcal{L}^v \phi(s,y) + l(y,v)\right] + g(z) - \phi(r,z) + \phi(t,x) \\ \text{ for all } (s,y,v,z) \in [t,r] \times \mathbb{R}^N \times U \times \mathbb{R}^N \end{array} \right\}$$

- with $\mathcal{L}^v \phi(s,y) = \langle f(y,v), D\phi(s,y)\rangle + \partial_t \phi(s,y)$

- The value function $V^r_{l,g}$ is the unique bounded, uniformly continuous viscosity solution of the equation

(HJB) $$\partial_t V^r_{l,g}(t,x) + H\left(t,x,DV^r_{l,g}(t,x)\right) = 0$$

for all $(t,x) \in (0,r) \times \mathbb{R}^N$, and $V^r_{l,g}(r,\cdot) = g(\cdot)$ on $\mathbb{R}^N$

- the Hamiltonian is given by

$$H(t,x,p) = \inf_{u \in U} \{\langle f(t,x,u), p\rangle + l(t,x,u)\}$$

for all $(t,x,p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$

- For proofs of the connection between $V^r_{l,g}$ and (HJB), see for instance Bardi & Capuzzo & Dolcetta '97 and the references therein

- For the notion of viscosity solutions see Lions '85, Dupuis & Ishii '90, Crandall & al. '92, Barles '94 (HJB of Neumann type) Barron & Jensen '90 and Frankowska '93 (discontinuous solutions)

Theorem (Main result for the Lipschitz case)

We have

$$V^r_{l,g} = \Lambda^r_{l,g} = \eta^r_{l,g}.$$

Consequently,

$$\Theta(t, r, x) = cl\ (co(\Gamma(t, r, x))).$$

Idea of the proof

- $\gamma^{t,r,x,u} \in \Gamma(t,r,x) \subset \Theta(t,r,x) \Longrightarrow V^r_{l,g}(t,x) \geq \Lambda^r_{l,g}(t,x)$
- $\eta \leq (r-t)\left[\mathcal{L}^v \phi(s,y) + l(s,y,v)\right] + g(z) - \phi(r,z) + \phi(t,x)$
  integrate w.r.t. $\gamma \in \Theta(t,r,x) \Longrightarrow \Lambda^r_{l,g}(t,x) \geq \eta^r_{l,g}(t,x)$
- approximate $V^r_{l,g}$ by smooth subsolutions V^ε
  $V^\varepsilon(t,x) - C\varepsilon \leq \eta^r_{l,g}(t,x)$ then $\varepsilon \to 0$ to get $\eta^r_{l,g}(t,x) \geq V^r_{l,g}(t,x)$

Proposition (Krylov)

There exists a constant $C > 0$ such that, for every $\varepsilon \in (0, 1]$, there exists $V^\varepsilon \in C_b^{1,2}\left([0, r+\varepsilon] \times \mathbb{R}^N\right)$ subsolution of (HJB) defined on $[0, r+\varepsilon] \times \mathbb{R}^N$ satisfying

$$(i)\left|V^\varepsilon(s,\cdot) - g(\cdot)\right| \leq C\varepsilon, \text{ for } s \in [r, r+\varepsilon] \text{ and}$$
$$(ii)\left|V^\varepsilon(\cdot) - V_{l,g}^r(\cdot)\right| \leq C\varepsilon, \text{ on } [0, r] \times \mathbb{R}^N.$$

- Let $l \equiv 0$ and denote by $\Theta(t, T, \cdot) := \Theta(t, \cdot)$, $\eta_{l,g}^T := \eta_g$, $\Lambda_{l,g}^T := \Lambda_g$ and $V_{l,g}^T := V_g$
- For all $(t, x) \in [0, T] \times \mathbb{R}^N$

$$D_g(t,x) = \left\{ \begin{array}{l} (\eta, \phi) \in \mathbb{R} \times C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right) \text{ s.t.} \\ \eta = \inf_{(s,y,v,z) \in [t,T] \times \mathbb{R}^N \times U \times \mathbb{R}^N} \{(T-t)\mathcal{L}^v \phi(s,y) + \\ g(z) - \phi(T,z) + \phi(t,x)\} \end{array} \right\}$$

- Note that $\Theta(t, x)$ contains measures with compact support

- The dual formulation becomes

$$V_g(t,x) = \sup\{\eta, (\eta,\phi) \in D_g(t,x)\}.$$

Definition

We say that $(\bar{\eta},\bar{\phi}) \in D_g(t,x)$ is an optimal pair if we have $V_g(t,x) = \bar{\eta}$.

- We denote by

$$\Omega_g(t,x) = \left\{ \begin{array}{l} (s,y,v,z) \in [t,r] \times \mathbb{R}^N \times U \times \mathbb{R}^N, \text{s.t.} \\ \bar{\eta} = (T-t)\mathcal{L}^v\bar{\phi}(s,y) + g(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x) \\ \text{for an optimal pair } (\bar{\eta},\bar{\phi}) \in D_g(t,x) \end{array} \right\}$$

- It is not always sure that optimal pairs exist. In this case we can define the set Ω using the sets $\Omega_{\frac{1}{n}}$ associated to a sequence of pairs $\left(\eta_{\frac{1}{n}}, \phi_{\frac{1}{n}}\right) \in D_{g'}(t,x)$ having the property that $\eta_{\frac{1}{n}} \nearrow \eta(t,x) = V(t,x)$. More precisely, the set Ω^c coincides with the $\limsup_{\frac{1}{n}\to 0} \Omega^c_{\frac{1}{n}}$.

Proposition

Let $(t,x) \in [0,T] \times \mathbb{R}^N$ *be fixed. Then,* $\gamma \in \Theta(t,x)$ *is optimal for* $\Lambda_g(t,x)$ *iff* $\gamma(\Omega_g(t,x)) = 1$.

- $\gamma \in \Theta(t,x)$ s.t. $\gamma\left(\Omega_g(t,x)\right) = 1$
- By definition,

$$\bar{\eta} = (T-t)\,\mathcal{L}^v\bar{\phi}(s,y) + g(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)$$

on $\Omega_g(t,x)$ for $(\bar{\eta},\bar{\phi}) \in D_g(t,x)$

- $\gamma \in \Theta(t,x)$, consequently,

$$\int_{\Omega_g(t,x)} \bar{\eta}\gamma(dsdydudz) = \int_{\Omega_g(t,x)} g\gamma(dsdydudz)$$
$$= \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} g(z)\,\gamma(dsdydudz)$$

- It follows that

$$\gamma(\Omega(t,x))\,\bar{\eta} = V(t,x) = \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} g(z)\,\gamma(dsdydudz)$$

and that $\gamma \in \Theta(t,x)$ is optimal

- Conversely, let $\gamma \in \Theta(t,x)$ be optimal.

$$
\begin{aligned}
&V(t,x) = \int\limits_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} g(z)\,\gamma(dsdydudz) = \\
&\int\limits_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left[(T-t)\,\mathcal{L}^v\bar{\phi}(s,y) + g(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right]\gamma(dsdydudz) = \\
&\int\limits_{\Omega_g(t,x)} \left[(T-t)\,\mathcal{L}^v\bar{\phi}(s,y) + g(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right]\gamma(dsdydudz) + \\
&\int\limits_{\Omega_g^c(t,x)} \left[(T-t)\,\mathcal{L}^v\bar{\phi}(s,y) + g(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right]\gamma(dsdydudz) > \\
&\bar{\eta} = V(t,x).
\end{aligned}
$$

for all optimal pairs $(\bar{\eta},\bar{\phi}) \in D_g(t,x)$

- Consequently, $\gamma\left(\Omega_g(t,x)\right) = 1$

$$\begin{cases} dX_s^{t,x,u} = f\left(X_s^{t,x,u}, u_s\right) ds, \ t_0 \leq t \leq T \\ X_t^{t,x,u} = x \in \mathbb{R}^N \end{cases}$$

- $r := T > 0$ finite time horizon, $t \in [0, T]$ and $l \equiv 0$
- Recall that $V(t,x) = \inf_{u \in \mathcal{U}} g\left(x_T^{t,x,u}\right)$
- Reachable set $R(T,t)x = \left\{x_T^{t,x,u} : u \in \mathcal{U}\right\}$
- We define $\overline{V}(t,x) = \inf\{g(x) : x \in cl(R(T,t)x)\}$
- Note that $\overline{V} = \Lambda_g^T = \eta_g^T$
- if f is convex in u Plaskacz, Quincampoix '01, S. '02, $V = \overline{V}$
- if g is u.s.c. $V = \overline{V}$

- $\mathbb{R}^2, U = \{-1, 1\}$
- $f : \mathbb{R}^3 \times U \longrightarrow \mathbb{R}$

$$f(x, y, u) = \left(u, x^2 \wedge 1\right)$$

$\forall t, x, y \in \mathbb{R}, u \in U$

- $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ l. s. c. defined by

$$g(x, y) = \begin{cases} 1, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

- $(0, 0) \in cl\,(R(T, t)(0, 0))$ and $(0, 0) \notin R(T, t)(0, 0) \Longrightarrow$

$$V = \inf_{u \in \mathcal{U}} h\left(x_T^{t,0,0,u}, y_T^{t,0,0,u}\right) = 1 \neq \overline{V}(t, 0, 0)$$

Theorem (Main result)

- *(a) g l.s.c., $\overline{V}$ is the smallest l.s.c. supersolution of* (HJB) *s.t.* $V(T,\cdot) \geq g(\cdot)$.
- *(b) g u.s.c., $\overline{V} = V$ is the largest u.s.c. subsolution of* (HJB) *s.t.* $V(T,\cdot) \leq g(\cdot)$.
- *(c) g is bounded*

$$\overline{V} = \inf \left\{ \begin{array}{c} \varphi : \varphi \textit{ l.s.c. subsolution of } \text{(HJB)} \textit{ s.t.} \\ \varphi(T,\cdot) \geq g(\cdot) \end{array} \right\} \textit{ and}$$

$$\overline{V} = \sup \left\{ \begin{array}{c} \varphi : \varphi \textit{ u.s.c. subsolution of } \text{(HJB)} \textit{ s.t.} \\ \varphi(T,\cdot) \leq g(\cdot) \end{array} \right\}.$$

We consider the control system:

$$\begin{cases} dX_s^{t,x,u} = b\left(X_s^{t,x,u}, u_s\right) ds + \sigma\left(X_s^{t,x,u}, u_s\right) dB_s, \ s \in [t, T], \\ X_t^{t,x,u} = x \in \mathbb{R}^N \text{ and } t \in [0, T] \end{cases}$$

Then, the value function is

$$V_h(t, x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[g\left(X_T^{t,x,u}\right)\right],$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a given function.

For any $X_\cdot^{t,x,u}$, we associate the occupational measure

$$\gamma^{t,x,u} = \left(\gamma_1^{t,x,u}, \gamma_2^{t,x,u}\right) \in \mathcal{P}\left([0, T] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$$

where

$$\begin{cases} \gamma_1^{t,x,u}(A \times B \times C) := \frac{1}{T-t}\mathbb{E}\left[\int_t^T 1_{A\times B\times C}\left(s, X_s^{t,x,u}, u_s\right) ds\right] \\ \gamma_2^{t,x,u}(D) := \mathbb{E}\left[1_D\left(X_T^{t,x,u}\right)\right] \end{cases}$$

for all Borel subsets $A \times B \times C \times D \subset [0,T] \times \mathbb{R}^N \times U \times \mathbb{R}^N$.

We obtain that

$$V(t,x) = \inf_{\gamma \in \Theta(t,x)} \int_{\mathbb{R}^N} g(z)\,\gamma\left([t,T], \mathbb{R}^N, U, dz\right)$$

The convex compact set $\Theta(t,x)$ is dependig only of b and σ.

Theorem

If g is Lipschitz and bounded then V_g is the unique viscosity solution of (HJB) *and* $V_g = \Lambda_g = \eta_g$.

Linear programming tools: Stockbridge (90), Bhatt, Borkar ('96), Kurtz, Stockbridge ('98), Borkar, Gaitsgory ('05)

- $\Lambda_g(t,x) = \inf_{\gamma\in\Theta(t,x)} \int_{\mathbb{R}^N} g(z)\,\gamma\left([t,T],\mathbb{R}^N,U,dz\right)$

  $(t,x)\in[0,T)\times\mathbb{R}^N$ and $\Lambda_g(T,\cdot) = g(\cdot)$
- $g:\mathbb{R}^N \longrightarrow \mathbb{R}$ is a lower semicontinuous function
- $\exists c\in\mathbb{R}$ such that $c\left(|x|^2+1\right) \geq h(x) \geq -c$

Theorem

Λ_g is the smallest lower semicontinuous viscosity supersolution of (HJB) *and*

$$\Lambda_g = \eta_g.$$

- $\begin{cases} dX_s^{t,x} = 0, \text{ for } 0 \leq t \leq s \leq T = 1, \\ X_t^{t,x} = x \in \mathbb{R}. \end{cases}$

  $g(\cdot) = 1_{\{0\}}(\cdot)$

- V_g is the largest u.s.c. subsolution of

  $\begin{cases} -\partial_t V_g(t,x) = 0, \text{ for all } (t,x) \in (0,T) \times \mathbb{R} \\ V_g(1,\cdot) = g(\cdot) \text{ on } \mathbb{R} \end{cases}$

- $V_g(t,\cdot) = g(\cdot)$ for every $t \in (0,T]$

- In particular, $V_g\left(\frac{1}{2},0\right)=1$
- $\eta_g\left(\frac{1}{2},0\right)$

$$= \sup\left\{\begin{array}{c} \eta\in\mathbb{R}:\exists\phi\in C_b^{1,2}\left([0,1]\times\mathbb{R}\right) \\ \text{s.t. } \forall\left(s,y,z\right)\in\left[\frac{1}{2},1\right]\times\mathbb{R}^2, \\ \eta\le\frac{1}{2}\partial_t\phi\left(s,y\right)+g\left(z\right)-\phi\left(1,z\right)+\phi\left(\frac{1}{2},0\right) \end{array}\right\}$$

- $z=\varepsilon,\varepsilon\to 0$ to get $\eta_g\left(\frac{1}{2},0\right)\le 0<V_g\left(\frac{1}{2},0\right)$.

- We consider

$$(SP)\qquad \begin{cases} dX_s^{t,x,y,u} = f\left(X_s^{t,x,y,u}, Y_s^{t,x,y,u}, u_s\right) ds, \\ \varepsilon dY_s^{t,x,y,u} = g\left(X_s^{t,x,y,u}, Y_s^{t,x,y,u}, u_s\right) ds, \end{cases}$$

  for all $s \geq t$, where $(t, x, y) \in [0, \infty) \times \mathbb{R}^M \times \mathbb{R}^N$ and $\varepsilon > 0$.
- U compact metric space
- admissible control $u \in \mathcal{U}$: Lebesque-measurable, U-valued
- $f : \mathbb{R}^N \times \mathbb{R}^M \times U \longrightarrow \mathbb{R}^N$ and $g : \mathbb{R}^N \times \mathbb{R}^M \times U \longrightarrow \mathbb{R}^N$
- (i) f, g bounded, uniformly continuous
- (ii) $\exists c > 0$ s.t. $|f(x, u) - f(y, u)| \leq c|x - y|$ and $|g(x, u) - g(y, u)| \leq c|x - y|$, $\forall (x, y, u) \in \times\mathbb{R}^{2N} \times U$

- We denote by $\left(x_{(\cdot)}^{t,x,y,u;\varepsilon}, y_{(\cdot)}^{t,x,y,u;\varepsilon}\right)$ the solution of (SP) starting from $(t,x,y) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N$, for some $u \in \mathcal{U}$
- Let $h : \mathbb{R}^M \to \mathbb{R}$ be bounded
- The value function:

$$W_{\varepsilon,h}(t,x,y) = \inf_{u \in \mathcal{U}} h\left(x_T^{t,x,y,u;\varepsilon}\right)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N$

- We call X the ”slow” variable and Y the ”fast” variable.

- Study of the behavior of $W_{\varepsilon,h}$ when $\varepsilon \to 0$
- Tikhonov approach (if stability) Tichonov'52 and Veliov '97.
- The averaging method Gaitsgory'92 and Grammel '96.
- Survey papers Naidu '02 and Dmitriev & Kurina '06.
- The averaging method: relationships between the system (SP) and

(AS) $$dX_s^{t,x} \in F\left(X_s^{t,x}\right)$$

- The averaging method: only the behavior of the "slow" variable $X_{(\cdot)}^{t,x,y,u;\varepsilon}$

- $(W_{\varepsilon,h}(\cdot,\cdot,y))_{\varepsilon>0} \to W_{F,h}$ defined by

$$W_{F,h}(t,x) = \inf_{x_{(\cdot)}^{t,x} \in S_F(t,x)} h\left(x_T^{t,x}\right)$$

  whenever h is uniformly continuous and bounded

- $S_F(t,x)$ denotes the set of solutions of (AS) starting from $(t,x) \in [0,T] \times \mathbb{R}^M$
- $S_\varepsilon(t,x,y)$ denotes the set of solutions of (SP) starting from $(t,x,y) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N$
- $S_F(0,x) =: S_F(x)$ and $S_\varepsilon(0,x,y) =: S_\varepsilon(x,y)$

- $\tau = \frac{s}{\varepsilon}$ in the system (SP) with $t = 0$
- $\left(\widetilde{X}_\tau, \widetilde{Y}_\tau, \widetilde{U}_\tau\right) = (X_{\varepsilon\tau}, Y_{\varepsilon\tau}, u_{\varepsilon\tau})$ for $\tau \in [0, \frac{T}{\varepsilon}]$, one gets

$$\begin{cases} d\widetilde{X}_\tau^{t,x,y,u} = \varepsilon f\left(\widetilde{X}_\tau^{t,x,y,u}, \widetilde{Y}_\tau^{t,x,y,u}, \widetilde{U}_\tau\right) d\tau, \\ d\widetilde{Y}_s^{t,x,y,u} = g\left(\widetilde{X}_\tau^{t,x,y,u}, \widetilde{Y}_\tau^{t,x,y,u}, \widetilde{U}_\tau\right) d\tau. \end{cases}$$

- Let x fixed in $\mathbb{R}^M$. The associated system:

(AS) $$dY_s^{t,y,u} = g\left(x, Y_s^{t,y,u}, u_s\right) ds,$$

- Denote by $y_{(\cdot)}^{t,y,u;x}$ the unique solution of (AS)

- The averaging method

$$A(x,y,S,u) = \frac{1}{S}\int_0^S f\left(x,\, Y_s^{t,y,u;x},\, u_s\right) ds$$

$$F(x,y,S) = \{A(x,y,S,u);\ u \in \mathcal{U}\}$$

- If controllability or stability of (AS) then the set $F(x,y,S) \to F(x)$ w. r. t. the Hausdorff metric and $F(x)$ is a compact convex set of $\mathbb{R}^M$

- Hypotheses

$$\begin{cases} \forall R > 0,\ \text{there exist nonempty bounded subsets } N_R \text{ and } \Omega_R \text{ of } \mathbb{R}^N \text{ s. t.} \\ \mathbf{1})\forall (x,y) \in B(0,R) \times N_R,\ \forall \left(x_{(\cdot)}^{0,x,y,u;\varepsilon}, y_{(\cdot)}^{0,x,y,u;\varepsilon}\right) \in S_\varepsilon(0,x,y) \\ y_s^{0,x,y,u;\varepsilon} \in \Omega_R,\ \text{for all } s \in [0,T] \text{ and all } \varepsilon > 0 \\ \mathbf{2})\forall (x,y) \in B(0,R) \times N_R,\ \forall\ u \in \mathcal{U},\ y_s^{0,y,u;x} \in \Omega_R,\ \text{for all } s \geq 0 \end{cases}$$

- Total controllability

$$(TC) \quad \begin{cases} \forall R > 0, \ \exists \ t_R \geq 0 \text{ such that } : \forall x \in B(0,R), \ \forall y_1, y_2 \in \Omega_R, \\ \text{there exists } u \in \mathcal{U} \text{ and } s \leq t_R \text{ such that } y_s^{0,y_2,u;x} = y_1. \end{cases}$$

- Stability property

$$(S) \quad \begin{cases} \forall R > 0, \ \text{there exists } \xi_R \in L^1([0,+\infty[,\mathbb{R}^+) \text{ such that :} \\ \forall x \in B(0,R), \ \forall y_1, y_2 \in \Omega_R, \forall u \in \mathcal{U}, \\ \left| y_s^{t,y_1,u;x} - y_s^{t,y_2,u;x} \right| \leq \xi_R(s) \left| y_1 - y_2 \right| \text{ for every } s \geq 0. \end{cases}$$

- If there exists C_R such that

$$\langle y_2 - y_1, g(x, y_2, u) - g(x, y_1, u) \rangle \leq -C_R \left| y_2 - y_1 \right|^2,$$

for every $u \in U$, then (S) is satisfied with $\xi_R(\tau) = e^{-C_R \tau}$

Proposition

We assume that F is locally Lipschitz. Then, for every $R_0 > 0$, every $R \geq R_0 + T\,|f|_\infty$, there exists a function $\mu_R : (0, +\infty) \to \mathbb{R}$, such that $\lim_{\varepsilon\to 0} \mu_R(\varepsilon) = 0$ and, for every $\varepsilon > 0$ and every $(x, y) \in B(0, R_0) \times N_R$:

-for every $\left(x_{(\cdot)}^{t,x,y,u_\varepsilon;\varepsilon}, y_{(\cdot)}^{t,x,y,u_\varepsilon;\varepsilon}\right) \in S_\varepsilon\,(t, x, y)$, there exists $x_{(\cdot)}^{t,x} \in S_F\,(t, x)$ and

(*) $$\sup_{s\in[t,T]} \left| x_s^{t,x,y,u_\varepsilon;\varepsilon} - x_s^{t,x} \right| \leq \mu_R(\varepsilon);$$

-for any $x_{(\cdot)}^{t,x} \in S_F\,(t, x)$, there exists

$(x_\varepsilon(\cdot), y_\varepsilon(\cdot)) \in \left(x_{(\cdot)}^{t,x,y,u_\varepsilon;\varepsilon}, y_{(\cdot)}^{t,x,y,u_\varepsilon;\varepsilon}\right) \in S_\varepsilon\,(t, x, y)$, such that (*) holds.

- Equivalently we have

$$\lim_{\varepsilon\to 0} d_H(S_\varepsilon\,(t, x, y)\,, S_F\,(t, x) \times \{0\}) = 0$$

- $T > 0$ We fix $T > 0$, $\varepsilon > 0$, $t \geq 0$, $(x_0, y_0) \in \mathbb{R}^M \times \mathbb{R}^N$ and $u \in \mathcal{U}$
- $\gamma^{t,T,x_0,y_0,u;\varepsilon} = \left(\gamma_1^{t,T,x_0,y_0,u;\varepsilon}, \gamma_2^{t,T,x_0,y_0,u;\varepsilon}\right)$

  $\in \mathcal{P}\left([t, T] \times \mathbb{R}^M \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^M \times \mathbb{R}^N\right)$

  $$\begin{cases} \gamma_1^{t,T,x_0,y_0,u;\varepsilon}(A \times B \times C \times D) = \frac{1}{T-t}\int_t^T 1_{A\times B\times C\times D}\left(s, X_s^{t,x_0,y_0,u;\varepsilon}, Y_s^{t,x_0,y_0,u;\varepsilon}, u\right. \\ \gamma_2^{t,T,x_0,y_0,u;\varepsilon} = \delta_{X_T^{t,x_0,y_0,u;\varepsilon}, Y_T^{t,x_0,y_0,u;\varepsilon}}, \end{cases}$$

  for all Borel sets $A \subset [t, T]$, $B \subset \mathbb{R}^M$, $C \subset \mathbb{R}^N$ and $D \subset U$.
- $\gamma^{t,t,x_0,y_0,u;\varepsilon} = \left(\gamma_1^{t,t,x_0,y_0,u;\varepsilon}, \gamma_2^{t,t,x_0,y_0,u;\varepsilon}\right)$

  $\in \mathcal{P}\left(\{t\} \times \mathbb{R}^M \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^M \times \mathbb{R}^N\right)$

  $$\gamma_1^{t,t,x_0,y_0,u;\varepsilon} = \delta_{t,x_0,y_0,u_t}, \gamma_2^{t,t,x_0,y_0,u;\varepsilon} = \delta_{x_0,y_0}.$$

$$\Gamma\left(t,x_0,y_0;\varepsilon\right)=\left\{\ \left(\gamma_1^{t,T,x_0,y_0,u;\varepsilon},\gamma_2^{t,T,x_0,y_0,u;\varepsilon}\right),\ \text{for all } u\in\mathcal{U}\ \right\}$$

$$\Theta\left(t,x_0,y_0;\varepsilon\right)=\left\{\begin{array}{l}\left(\gamma_1,\gamma_2\right)\in\mathcal{P}\left(\left[t,T\right]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right)\\ \forall\phi\in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right),\\ \int\limits_{[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left(\phi\left(t,x_0,y_0\right)+\left(T-t\right)\mathcal{L}^{u;\varepsilon}\phi\left(s,x,y\right)\right.\\ \left.-\phi\left(T,z,w\right)\right)\gamma_1\left(dsdxdydu\right)\gamma_2\left(dzdw\right)=0\end{array}\right\}$$

where

$$\mathcal{L}^{u;\varepsilon}\phi\left(s,x,y\right)=\left\langle\left(f\left(x,y,u\right),\frac{1}{\varepsilon}g\left(x,y,u\right)\right),D\phi\left(s,x,y\right)\right\rangle+\partial_t\phi\left(s,x,y\right)$$

- The linearized problem

$$\Lambda_{\varepsilon,h}(t,x_0,y_0) = \inf_{\gamma=(\gamma_1,\gamma_2)\in\Theta(t,x_0,y_0;\varepsilon)} \int_{\mathbb{R}^M\times\mathbb{R}^N} h(z)\,\gamma_2(dzdw)$$

- The dual formulation

$\eta_{\varepsilon,h}(t,x_0,y_0) =$

$$\sup\left\{\begin{array}{l} \eta\in\mathbb{R} : \exists\phi\in C^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right) \text{ s.t.} \\ \forall(s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}\times\mathbb{R}^N, \\ \eta\le(T-t)\,\mathcal{L}^{v;\varepsilon}\phi(s,x,y)+h(z)-\phi(T,z,w)+\phi(t,x_0,y_0) \end{array}\right\}$$

- For every $\varepsilon>0$ we have

$$W_{\varepsilon,h} = \Lambda_{\varepsilon,h} = \eta_{\varepsilon,h}$$

$$\Theta\left(t,x_0,y_0\right)=$$

$$\left\{\begin{array}{l}
\left(\gamma_1,\gamma_2\right)\in\mathcal{P}\left([t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right)\\
\forall\psi\in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\right)\text{ and }\forall\phi\in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right),\\
\int\limits_{[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left(\psi\left(t,x_0\right)+\left(T-t\right)\mathcal{L}^{u,f}\psi\left(s,x\right)\right.\\
\left.-\psi\left(T,z\right)\right)\gamma_1\left(dsdxdydu\right)\gamma_2\left(dzdw\right)=0\text{ and}\\
\int\limits_{[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\mathcal{L}^{u,g}\phi\left(s,x,y\right)\gamma_1\left(dsdxdydu\right)\gamma_2\left(dzdw\right)=0
\end{array}\right\}$$

where

$$\mathcal{L}^{u,f}\psi\left(s,x,y\right)=\left\langle f\left(x,y,u\right),D_x\psi\left(s,x\right)\right\rangle+\partial_t\psi\left(s,x\right)$$

$$\mathcal{L}^{u,g}\phi\left(s,x,y\right)=\left\langle g\left(x,y,u\right),D_y\phi\left(s,x,y\right)\right\rangle$$

- The linearized problem

$$\Lambda_h(t,x_0,y_0) = \inf_{\gamma=(\gamma_1,\gamma_2)\in\Theta(t,x_0,y_0)} \int_{\mathbb{R}^M\times\mathbb{R}^N} h(z)\,\gamma_2(dzdw)$$

- We denote by

$$\eta_h(t,x_0,y_0) =$$

$$\sup\left\{\begin{array}{l} \eta\in\mathbb{R} : \exists\alpha\in C(\mathbb{R}_+) \text{ with } \lim\limits_{\varepsilon\to 0}\alpha(\varepsilon)=0 \text{ s.t. } \forall\varepsilon>0, \\ \exists\phi\in C^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right) \text{ and } \psi\in C^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^M\right) \text{ s.t.} \\ \|\phi-\psi\|_\infty+\|\nabla\phi-\nabla\psi\|_\infty\le\alpha(\varepsilon) \text{ and s.t.} \\ \forall(s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}\times\mathbb{R}^N, \\ \eta\le(T-t)\,\mathcal{L}^{v;\varepsilon}\phi(s,x,y)+h(z)-\phi(T,z,w)+\phi(t,x_0,y_0) \end{array}\right\}$$

Theorem (Main result)

We have that

$$W_h(t,x_0) = \Lambda_h(t,x_0,y_0) = \eta_h(t,x_0,y_0).$$

for all $(t,x_0,y_0)\in[0,T]\times\mathbb{R}^M\times\mathbb{R}^N$.

- $\bar{\gamma}^{t,T,x_0,y_0;\varepsilon} = \left(\bar{\gamma}_1^{t,T,x_0,y_0;\varepsilon} \bar{\gamma}_2^{t,T,x_0,y_0;\varepsilon}\right) \in \Theta\left(t, x_0, y_0; \varepsilon\right)$ optimal, i.e.

$$\Lambda_{\varepsilon,h}\left(t, x_0, y_0\right) = \int_{\mathbb{R}^M \times \mathbb{R}^N} h\left(z\right) \bar{\gamma}_2^{t,T,x_0,y_0;\varepsilon}\left(dzdw\right)$$

- $\bar{\gamma}^{t,T,x_0,y_0;\varepsilon} \rightharpoonup \bar{\gamma} \in \Theta\left(t, x_0, y_0\right)$

- Consequently,

$$\Lambda_h\left(t, x_0, y_0\right) \leq \int_{\mathbb{R}^M \times \mathbb{R}^N} h\left(z\right) \gamma_2^{t,T,x_0,y_0}\left(dzdw\right) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^M \times \mathbb{R}^N} h\left(z\right) \gamma_2^{t,T,x_0,y_0;\varepsilon}\left(dzdw\right)$$

$$= \lim_{\varepsilon \to 0} \Lambda_{\varepsilon;h}\left(t, x_0, y_0\right) = \lim_{\varepsilon \to 0} W_{\varepsilon,h}\left(t, x_0, y_0\right) = W_h\left(t, x_0\right)$$

- Let $\gamma \in \Theta(t, x_0, y_0)$ and $\eta \in \mathbb{R}$ such that

$$\begin{aligned}
&\exists \alpha \in C(\mathbb{R}_+) \text{ with } \lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0 \text{ s.t. } \forall \varepsilon > 0, \\
&\exists \phi \in C^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right) \text{ and } \psi \in C^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^M\right) \text{ s.t.} \\
&\|\phi - \psi\|_\infty + \|\nabla \phi - \nabla \psi\|_\infty \leq \alpha(\varepsilon) \text{ and s.t.} \\
&\forall (s, x, y, v, z, w) \in [t, T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M \times \mathbb{R}^N, \\
&\eta \leq (T - t) \mathcal{L}^{v;\varepsilon} \phi(s, x, y) + h(z) - \phi(T, z, w) + \phi(t, x_0, y_0).
\end{aligned}$$

- By integrating with respect to γ we obtain that

$$\eta \leq \int_{\mathbb{R}^M \times \mathbb{R}^N} h(z)\, \gamma_2(dzdw)$$

and consequently,

$$\eta_h(t, x_0, y_0) \leq \int_{\mathbb{R}^M \times \mathbb{R}^N} h(z)\, \gamma_2(dzdw)$$

for all $\gamma \in \Theta(t, x_0, y_0)$. We have that $\eta_h(t, x_0, y_0) \leq \Lambda_h(t, x_0, y_0)$.

- There exist two families $V^{\delta_\varepsilon}_{\varepsilon,h} \in C^{1,1}_b\left([0, T+\delta] \times \mathbb{R}^M \times \mathbb{R}^N\right)$ and $V^{\delta_\varepsilon}_h \in C^{1,1}_b\left([0, T+\delta] \times \mathbb{R}^N\right)$ as in the definition of η s. t.

$$\forall (s, x, y, v, z, w) \in [t, T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M \times \mathbb{R}^N,$$

$$V^{\delta_\varepsilon}_{\varepsilon,h}(t, x_0, y_0) \le (T-t)\,\mathcal{L}^{v;\varepsilon} V^{\delta_\varepsilon}_{\varepsilon,h}(s, x, y) + h(z) - V^{\delta_\varepsilon}_{\varepsilon,h}(T, z, w) + V^{\delta_\varepsilon}_{\varepsilon,h}(t, x_0, .$$

Moreover $\lim_{\varepsilon\to 0} V^{\delta_\varepsilon}_{\varepsilon,h}(t, x_0, y_0) = W_h(t, x_0)$.

- By the definition of $\eta_h(t, x_0, y_0)$: $V^{\delta_\varepsilon}_{\varepsilon,h}(t, x_0, y_0) \le \eta(t, x_0, y_0)$.
- Consequently,

$$W_h(t, x_0) = \lim_{\varepsilon\to 0} V^{\delta_\varepsilon}_{\varepsilon,h}(t, x_0, y_0) \le \eta_h(t, x_0, y_0)$$

- When the perturbed system is fully nonlinear it is very difficult to characterize the optimal trajectories using the Pontryagin maximum principle because we do not know exactly the form of the averaged dynamics
- We denote by

$D_{\varepsilon,h}(t, x_0, y_0) =$

$$\left\{ \begin{array}{l} (\eta,\phi) \in \mathbb{R} \times C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right) \text{ s.t.} \\ \eta = \inf_{(s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N} \{(T-t)\,\mathcal{L}^{v;\varepsilon}\phi(s,x,y) \\ +h(z) - \phi(T,z,w) + \phi(t,x_0,y_0)\} \end{array} \right\}$$

for all $(t, x_0, y_0) \in [0, T] \times \mathbb{R}^M \times \mathbb{R}^N$.

- We have

$$W_{\varepsilon,h}(t,x_0,y_0) = \sup\{\eta, (\eta,\phi) \in D_{\varepsilon,h}(t,x_0,y_0)\}.$$

Definition

$(\bar{\eta}_\varepsilon, \bar{\phi}_\varepsilon) \in D_{\varepsilon,h}(t,x_0,y_0)$ is an optimal pair whenever $W_{\varepsilon,h}(t,x_0,y_0) = \bar{\eta}_\varepsilon$.

- We denote by

$\overline{\Omega}_{\varepsilon,h}(t,x_0,y_0) =$

$$\left\{\begin{array}{l} (s,x,y,v,z,w) \in [t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N \text{ s.t.} \\ \bar{\eta}_\varepsilon = \{(T-t)\,\mathcal{L}^{v;\varepsilon}\phi(s,x,y) + h(z) - \phi(T,z,w) + \phi(t,x_0,y_0)\} \\ \text{for an optimal pair } (\bar{\eta}_\varepsilon, \bar{\phi}_\varepsilon) \in D_{\varepsilon,h}(t,x_0,y_0) \\ \text{and with } \bar{\phi}_\varepsilon \text{ as in the definition of } \eta_h(t,x_0,y_0) \end{array}\right\}$$

- and by

$$\Omega^c_h(t,x_0,y_0) = \limsup_{n\to\infty} \Omega^c_{\frac{1}{n},h}(t,x_0,y_0).$$

- Note that, $\gamma_\varepsilon \in \Theta(t, x_0, y_0; \varepsilon)$ is optimal iff $\gamma_\varepsilon(\Omega_{\varepsilon,h}(t, x_0, y_0)) = 1$, $\varepsilon > 0$

Theorem (Main result)

For every $(t, x_0, y_0) \in [0, T] \times \mathbb{R}^M \times \mathbb{R}^N$*, we have that* $\gamma \in \Theta(t, x_0, y_0)$ *is optimal iff* $\gamma(\Omega_h(t, x_0, y_0)) = 1$*.*

Thank you for your attention!