Tropical spectrahedra and stochastic games

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Talk based on: X. Allamigeon, S. Gaubert, and M. Skomra. "Solving generic nonarchimedean semidefinite programs using stochastic game algorithms". arXiv:1603.06916. 2016

Section I: Motivation

Complexity issues in semidefinite programming

Spectrahedra

Definition (spectrahedron)

Given symmetric matrices $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$, the associated **spectrahedron** is defined as

$$\mathcal{S} = \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \dots + x_n Q^{(n)} \text{ is positive semidefinite}\} .$$

Example: an *elliptope* in $\mathbb{R}^3$ is a spectrahedron defined by

$$Q(x) = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} .$$

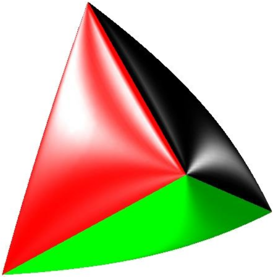

Figure: 3D elliptope. Source: `http://www.math.uni-frankfurt.de/~rostalsk/pmwiki`

Semidefinite programming (SDP)

Definition (semidefinite programming)

The task of minimizing a linear function over a spectrahedron is known as **semidefinite programming (SDP)**.

Complexity of SDP

- It is sometimes said that SDP is solvable in polynomial time, but this is true only in a restricted sense.[1,2]

[1] E. de Klerk and F. Vallentin. "On the Turing model complexity of interior point methods for semidefinite programming". arXiv:1507.03549. 2015.

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$$\mathrm{Poly}(n, m, \log \varepsilon, \log R, \log r, \dots),$$

where $(R, r, \dots)$ are some metric estimations of the spectrahedron (and can be very large).

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  where $(R, r, \dots)$ are some metric estimations of the spectrahedron (and can be very large).
- We suppose that the program has both primal and dual strongly feasible points or, at least, that it has no duality gap.

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Semidefinite feasibility problem

Definition (Semidefinite feasibility problem, SDFP)

Given symmetric matrices $Q^{(0)}, \dots, Q^{(n)}$, decide whether the associated spectrahedron $\mathcal{S}$ is nonempty.

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- Exact answers to SDFP can be obtained by quantifier elimination or critical points methods.[3,4]

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Motivating question

Is there a different approach to SDP?

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Section II: Our contribution

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- We show that the answer is positive... if one regards generic SDFPs over a nonarchimedean field.

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- SDP is valid over any real closed field.

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- SDP is valid over any real closed field.
- Main examples of real closed fields:
  - archimedean field of real numbers
  - nonarchimedean field of Puiseux series.

Our contribution

- We show that the answer is positive... if one regards generic SDFPs over a nonarchimedean field.
- SDP is valid over any real closed field.
- Main examples of real closed fields:
  - archimedean field of real numbers
  - nonarchimedean field of Puiseux series.

Theorem (in informal terms)

Generic instances of SDFP over the nonarchimedean field of Puiseux series can be solved efficiently using combinatorial algorithms for stochastic games.*
(*) Currently, the result is proven only for Metzler conic programs.

Puiseux series

- A (formal generalized) **Puiseux series** is a series of form

$$\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} \, ,$$

where the sequence $(\alpha_i)_i \subset \mathbb{R}$ is strictly decreasing and either finite or unbounded c_i are real.

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- The subset of absolutely converging (for t large enough) Puiseux series forms a real closed field[5], denoted here by $\mathbb{K}$.

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- The subset of absolutely converging (for t large enough) Puiseux series forms a real closed field[5], denoted here by $\mathbb{K}$.
- We say that $\boldsymbol{x} \geq \boldsymbol{y}$ if $\boldsymbol{x}(t) \geq \boldsymbol{y}(t)$ for all t large enough. This is a linear order on $\mathbb{K}$.

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SDFP over Puiseux series

Definition (SDFP over Puiseux series)

Given symmetric matrices $\boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$, denote

$$\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} .$$

We want to decide if the spectrahedron

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}_{\geq 0}^n \colon \boldsymbol{Q}(\boldsymbol{x}) \text{ is positive semidefinite}\}$$

is nonempty.

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Given symmetric matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(n)}$, denote

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We want to decide if the spectrahedron

$$\mathcal{S} = \{x \in \mathbb{K}^n_{\geq 0} \colon Q(x) \text{ is positive semidefinite}\}$$

is nonempty.

- By model completeness of real closed fields, SDFP over Puiseux series is at least as hard as SDFP over real numbers.
- Here, we restrict ourselves to the subclass of **generic** SDFPs over Puiseux series.

Example

Take the spectrahedral cone

$$Q(x) := \begin{bmatrix} tx_3 & -x_1 & -t^{3/4}x_3 \\ -x_1 & t^{-1}x_1 + t^{-5/4}x_3 - x_2 & -x_3 \\ -t^{3/4}x_3 & -x_3 & t^{9/4}x_2 \end{bmatrix} \succeq 0 \,.$$

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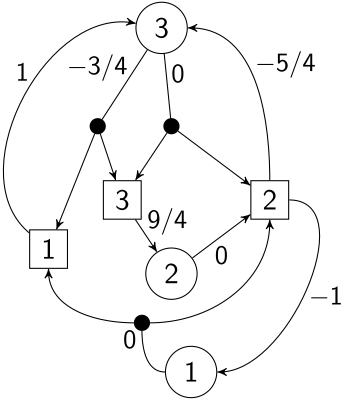
- We associate with $Q(x)$ a stochastic game with perfect information.



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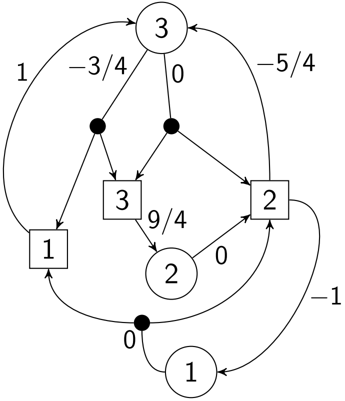
- We associate with $Q(\boldsymbol{x})$ a stochastic game with perfect information.
- Solving this game enables us to decide that the cone is nontrivial and to compute a feasible point $(t^{-1/16}, t^{-9/8}, 1)$.



Tropical geometry

- With every point $\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} \in \mathbb{K}$ we associate its **valuation** defined as the highest exponent occurring in $\boldsymbol{x}$,

$$\mathrm{val}(\boldsymbol{x}) = \lim_{t \to \infty} \log_t |\boldsymbol{x}(t)| = \alpha_1 \, .$$

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- We extend the valuation map to vectors by applying it coordinate-wise.

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Tropical geometry

Tropical geometry studies the images of algebraic sets over the field of Puiseux series by the valuation map.

Definition

Suppose that $\mathcal{S}$ is a spectrahedron in $\mathbb{K}^n_{\geq 0}$. Then we say that $\mathrm{val}(\mathcal{S})$ is a **tropical spectrahedron**.

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Tropical spectrahedra have polyhedral structure. This follows from general results of model theory.[6]

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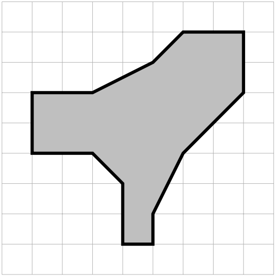

Figure: Tropical spectrahedron.

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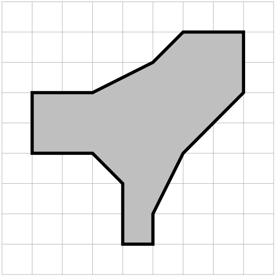

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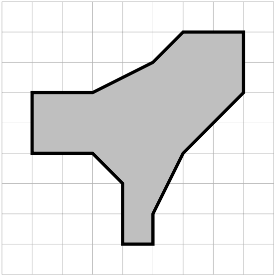
- Computing $\mathrm{val}(\mathcal{S})$ is difficult...
- but not if $\mathcal{S}$ is generic!
- In the generic case, $\mathrm{val}(\mathcal{S})$ can be computed out of 1×1 and 2×2 minors of $Q(x)$.



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Our assumptions - Metzler spectrahedral cones

Definition

A square matrix is called a **Metzler matrix** if its off-diagonal entries are nonpositive.

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Definition

A square matrix is called a **Metzler matrix** if its off-diagonal entries are nonpositive.

- We take Metzler matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$, let

$$\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \,,$$

  and define a spectrahedral cone

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geq 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \text{ is positive semidefinite}\} \,.$$

- We want to decide if this cone is nontrivial (i.e., if it contains a point different than 0).

Characterization of spectrahedra by minors

Lemma (inclusion of Metzler spectrahedra)

Let $\boldsymbol{\mathcal{S}}_2$ *denote the set of all* $\boldsymbol{x} \in \mathbb{K}^n_{\geq 0}$ *such that all principal* 1×1 *and* 2×2 *minors of* $\boldsymbol{Q}(\boldsymbol{x})$ *are nonnegative. Then* $\boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{S}}_2$*.*

Proof.

A matrix is positive semidefinite if and only if all of its principal minors are nonnegative. □

Tropical Metzler spectrahedra

Theorem (tropical Metzler spectrahedra)

For tropically generic Metzler matrices $(\boldsymbol{Q}^{(k)})_k$ *the set* $\mathrm{val}(\boldsymbol{\mathcal{S}})$ *is described by the tropical minor inequalities of order* 1 *and* 2,

$$\forall i, \max_{\boldsymbol{Q}^{(k)}_{ii} > 0} (x_k + \mathrm{val}(\boldsymbol{Q}^{(k)}_{ii})) \geq \max_{\boldsymbol{Q}^{(l)}_{jj} < 0} (x_l + \mathrm{val}(\boldsymbol{Q}^{(l)}_{jj}))$$

and

$$\forall i \neq j, \max_{\boldsymbol{Q}^{(k)}_{ii} > 0} (x_k + \mathrm{val}(\boldsymbol{Q}^{(k)}_{ii})) + \max_{\boldsymbol{Q}^{(k)}_{jj} > 0} (x_k + \mathrm{val}(\boldsymbol{Q}^{(k)}_{jj})) \geq 2 \max_{\boldsymbol{Q}^{(l)}_{ij} < 0} (x_l + \mathrm{val}(\boldsymbol{Q}^{(l)}_{ij})) \, .$$

Shapley operators

Lemma

The set $\mathrm{val}(\boldsymbol{\mathcal{S}})$ *can be equivalently defined as the set of all* x *such that for all* k *we have*

$$x_k \leq \min_{Q_{ij}^{(k)}<0} \Big(-\mathrm{val}(Q_{ij}^{(k)}) + \frac{1}{2}\big(\max_{Q_{ii}^{(l)}>0} (\mathrm{val}(Q_{ii}^{(l)}) + x_l) + \max_{Q_{jj}^{(l)}>0} (\mathrm{val}(Q_{jj}^{(l)}) + x_l)\big)\Big) \, .$$

In other words, we have

$$\mathrm{val}(\boldsymbol{\mathcal{S}}) = \{x \in (\mathbb{R} \cup \{-\infty\})^n \colon x \leq F(x)\} \, ,$$

where F *is a Shapley operator of a stochastic game. We denote this game by* Γ.

Collatz-Wielandt property

- If Γ is a stochastic mean payoff game in which Player Max chooses the initial state, then we have

$$v = \max_{\lambda}\{x \in (\mathbb{R} \cup \{-\infty\})^n \colon x + \lambda e \leq F(x), x \neq -\infty\} \, ,$$

where v denotes the value of Γ and F is its Shapley operator.

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where v denotes the value of Γ and F is its Shapley operator.

- This follows from
  - M. Akian, S. Gaubert, and A. Guterman. "Tropical polyhedra are equivalent to mean payoff games". In: *Int. J. Algebra Comput.* 22.1 (2012),

where it was used to establish the correspondence between **deterministic** mean payoff games and nonarchimedean **linear** programming.

Application to SDFP over Puiseux series

Assumption (Subclass of SDFP over Puiseux series)

Given tropically generic symmetric Metzler matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ *we want to decide if the spectrahedral cone*

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geq 0} \colon \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \text{ is positive semidefinite}\}$$

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is nontrivial.

Theorem

We can construct a stochastic mean payoff game Γ *such that its value is nonnegative if and only if* $\boldsymbol{\mathcal{S}}$ *is nontrivial.*

Example

$$Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix},$$

$$Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}.$$

Construction of Γ

We construct Γ as follows:

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Construction of Γ

The number of matrices (here: 3) defines the number of states controlled by Player Min.

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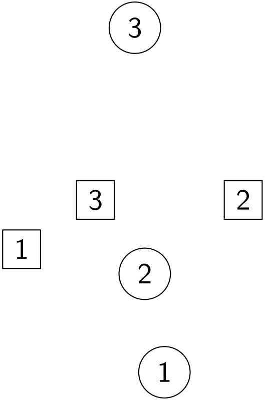

Construction of Γ

The size of matrices (here: 3×3) defines the number of states controlled by Player Max (here: 3).

Example

$$Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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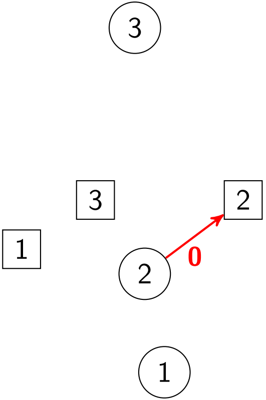


Construction of Γ

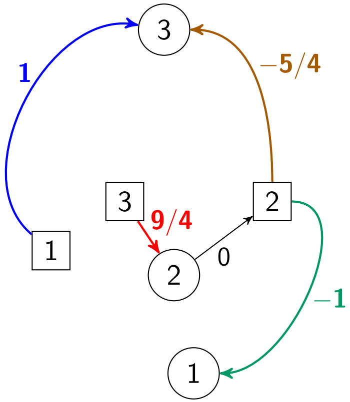
If $Q^{(k)}_{ii}$ is negative, then Player Min can move from state k to state i. After this move Player Max receives $-\mathrm{val}(Q^{(k)}_{ii})$.

Example

$$Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & \mathbf{t}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathbf{t}^{9/4} \end{bmatrix},$$

$$Q^{(3)} := \begin{bmatrix} \mathbf{t} & 0 & -t^{3/4} \\ 0 & \mathbf{t}^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}.$$



Construction of Γ

If $Q^{(k)}_{ii}$ is positive, then Player Max can move from state i to state k. After this move Player Max receives $\mathrm{val}(Q^{(k)}_{ii})$.

Example

$$Q^{(1)} := \begin{bmatrix} 0 & -\mathbf{1} & 0 \\ -\mathbf{1} & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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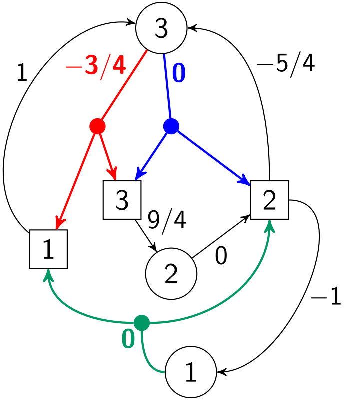


Construction of Γ

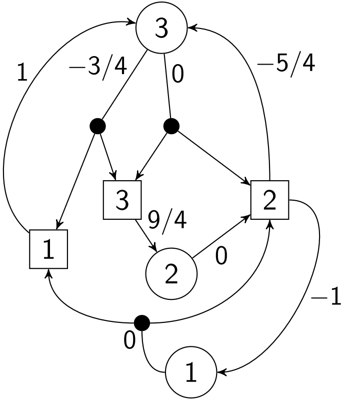
If $Q^{(k)}_{ij}$ is nonzero, $i \neq j$, then Player Min have a coin-toss move from state k to states (i, j) and Player Max receives $-\mathrm{val}(Q^{(k)}_{ij})$.

Example

There is only one pair of optimal policies

$$\textcircled{3} \to \left\{ \boxed{1}, \boxed{3} \right\},$$

$$\boxed{2} \to \textcircled{1}.$$



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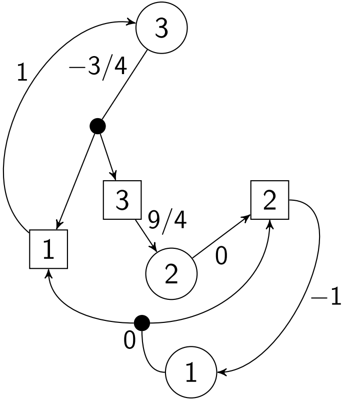

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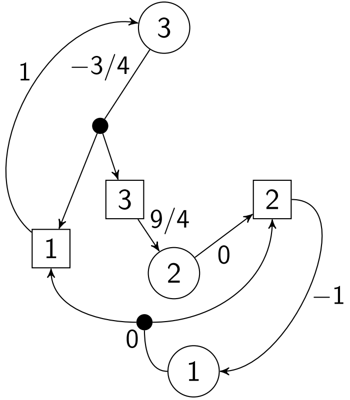


Corollary

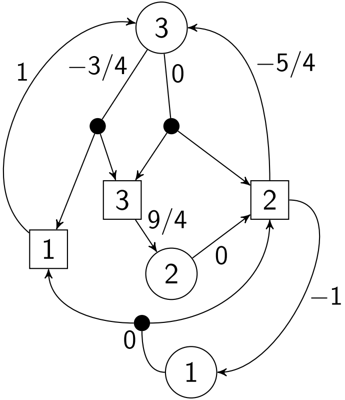
The spectrahedral cone $\boldsymbol{\mathcal{S}}$ has a nontrivial point in the positive orthant $\mathbb{K}^3_{\geq 0}$.

Example

The Shapley operator is given by

$$F(x) = \left(\frac{x_1 + x_3}{2}, x_1 - 1, \frac{x_2 + x_3}{2} + \frac{7}{8}\right)$$

and $(-\frac{1}{16}, -\frac{9}{8}, 0)$ is a bias vector.



Corollary

The spectrahedral cone $\mathcal{S}$ has a nontrivial point in the positive orthant $\mathbb{K}^3_{\geq 0}$. For example, it contains the point $(t^{-1/16}, t^{-9/8}, 1)$.

Benchmark

We tested our method on randomly chosen matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ with positive entries on diagonals and no zero entries. We used the value iteration algorithm.

(n, m)	$(10, 8)$	$(10, 10)$	$(10, 100)$	$(10, 500)$	$(10, 1000)$	$(10, 2000)$
time	0.007	0.009	0.009	0.042	0.159	0.564
(n, m)	$(50, 10)$	$(50, 40)$	$(50, 45)$	$(50, 50)$	$(50, 100)$	$(50, 1000)$
time	0.008	0.055	0.023	0.015	0.015	0.778
(n, m)	$(100, 10)$	$(100, 80)$	$(100, 90)$	$(100, 95)$	$(100, 100)$	$(100, 500)$
time	0.009	0.156	0.053	0.091	0.040	0.362
(n, m)	$(1000, 10)$	$(1000, 100)$	$(1000, 200)$	$(2000, 10)$	$(2000, 50)$	$(2000, 100)$
time	0.037	0.480	1.437	0.098	0.535	1.29

Table: Execution time (in sec.) of Procedure CHECKFEASIBILITY on random instances.

Concluding remarks

- We study the nonarchimedean version of semidefinite programming and show a correspondence

Nonarchimedean SDP $\longleftrightarrow$ Stochastic mean payoff games

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Concluding remarks

- We study the nonarchimedean version of semidefinite programming and show a correspondence

Nonarchimedean SDP $\longleftrightarrow$ Stochastic mean payoff games

- This leads to an algorithm for generic semidefinite feasibility problems over Puiseux series.
- We relate two problems of open complexity:
  - Semidefinite feasibility problem (not known to be in **NP**)
  - Stochastic mean payoff games (belongs to **NP** $\cap$ **coNP**, not known to be in **P**).

Further study

- Generalization to the non-Metzler, nonconic case.

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- Generalization to the non-Metzler, nonconic case.
- Replacing the formal parameter t by a real number. Can this method lead to an algorithm for SDP in real numbers?
- Combinatorial description of **all** tropical spectrahedra (not only the generic ones).

Thank you for your attention

X. Allamigeon, S. Gaubert, and M. Skomra. "Solving generic nonarchimedean semidefinite programs using stochastic game algorithms". arXiv:1603.06916. 2016

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