Lipschitz continuity for solutions of Hamilton-Jacobi equation with Ornstein-Uhlenbeck operator

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March 23, 2016

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- Statement and application of Lipschitz regularity result.
- Proof of the Lipschitz regularity theorem.

We consider the stochastic differential equation

$$\begin{cases} dX_t = (g(X_t, \gamma_t) - \alpha X_t)dt + 2\sigma(X_t)dW_t \\ X_0 = x \in \mathbb{R}^N, \end{cases}$$

(W_t) is an N-dimentional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, the control γ_t takes value on a compact set $A \subset \mathbb{R}^N$ which is defined as the collection of all $\mathcal{F}_t$-progressively measurable, g is a continuous vector on $\mathbb{R}^N$, σ is a continuous $N \times N$ matrix and α-terms is called Ornstein-Uhlenbeck operator ($\alpha > 0$).

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Value functions

$$\text{infinite horizon: } u^\lambda(x) = \inf_\gamma \mathbb{E}\left(\int_0^\infty e^{-\lambda s} L(X_s, \gamma_s)ds\right), \quad (1)$$

$$\text{finite horizon: } u(x,t) = \inf_\gamma \mathbb{E}\left(\int_0^t L(X_s, \gamma_s)ds\right), \quad (2)$$

where L is a running cost or Lagrangian function defined $L(X_s, \gamma_s) = l(X_s, \gamma_x) + f(X_s)$, $l(x, .)$ is a bounded function in x and f is possibly an unbounded function.

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where L is a running cost or Lagrangian function defined $L(X_s,\gamma_s) = l(X_s,\gamma_x) + f(X_s)$, $l(x,.)$ is a bounded function in x and f is possibly an unbounded function.
By the Dynamic Programming Principle, $u^\lambda(x)$, $u(x,t)$ are viscosity solution of

$$\lambda u^\lambda(x) - \mathrm{tr}(\sigma(x)\sigma^T(x)D^2u^\lambda(x)) + \alpha x.Du^\lambda(x) + H(x, Du^\lambda(x)) = f(x) \quad \text{in } \mathbb{R}^N$$

$$u_t(x,t) - \mathrm{tr}(\sigma(x)\sigma^T(x)D^2u(x,t)) + \alpha x.Du(x,t) + H(x,Du(x,t)) = f(x) \qquad \textbf{(CP)}$$

respectively, where

$$H(x,p) = \max_{a\in A}\{-g(x,a).p - l(x,a)\}. \tag{3}$$

Motivations

By letting $\lambda \to 0$ in (1) or $t \to \infty$ in (2), it is called "ergodic control problem".
Question: Under which conditions on (g, σ),

$$\text{Question 1: } \lambda u^{\lambda} \to ? \text{ as } \lambda \to 0,$$

$$\text{Question 2: } \frac{u(., t)}{t} \to ? \text{ as } t \to \infty.$$

Some references: Arisawa(1997), Arisawa-Lions (1998), Lions-Papanicolaou-Varadhan, Fujita-Ishii-Loreti(2006),...

Stationary problem

For $\lambda \in (0,1)$, we consider the stationary problem

$$\lambda u^\lambda(x) - \mathrm{tr}(\sigma(x)\sigma^T(x)D^2u^\lambda(x)) + \alpha x.Du^\lambda(x) + H(x, Du^\lambda(x)) = f(x) \quad \text{in } \mathbb{R}^N. \qquad \textbf{(SP)}$$

Here σ is a diffusion matrix.
Let $\mu > 0$ we define the function $\phi_\mu \in C^\infty(\mathbb{R}^N)$ by

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}} \text{ for } x \in \mathbb{R}^N.$$

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We introduce the class of solution for **(SP)**

$$\mathcal{E}_\mu(\mathbb{R}^N) = \{v : \mathbb{R}^N \to \mathbb{R} : \lim_{|x|\to+\infty} \frac{v(x)}{\phi_\mu(x)} = 0\}.$$

Hereafter we use ϕ instead of ϕ_μ for simplicity of notation.

Assumptions on the datas

- Diffusion matrix:

$$\begin{cases} \sigma \in C(\mathbb{R}^N, \mathcal{M}_N), \text{there exists } C_\sigma,\ L_\sigma > 0 \text{ such that} \\ |\sigma(x)| \leq C_\sigma, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y| \quad x \in \mathbb{R}^N. \end{cases} \tag{4}$$

$$\begin{cases} \text{(ellipticity) There exists } \nu > 0 \text{ such that} \\ \nu I \leq \sigma(x)\sigma(x)^T \quad x \in \mathbb{R}^N, \end{cases} \tag{5}$$

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- Assumption on H:

$$|H(x,p)| \leq C_H(1 + |p|), \quad x, p \in \mathbb{R}^N. \tag{6}$$

This is very general assumption for the Hamiltonian H. From the definition H in (3), it is enough to make sure that g and l are bounded in x.

- Assumption on f:

$$\begin{cases} \text{For } f \in \mathcal{E}_\mu(\mathbb{R}^N), \text{there exists } C_f, \mu > 0 \text{ such that} \\ |f(x) - f(y)| \leq C_f(\phi_\mu(x) + \phi_\mu(y))|x-y|, \quad x, y \in \mathbb{R}^N. \end{cases} \tag{7}$$

Statement of Lipschitz regularity result

Theorem 1

Let $\mu > 0$, $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ be a solution of **(SP)**. *Assume that (4), (5), (6) and (7) hold. For any $\alpha > 0$, there exists a constant $C > 0$* **independent** *of λ such that*

$$|u^\lambda(x) - u^\lambda(y)| \le C|x-y|(\phi(x)+\phi(y)), \quad x, y \in \mathbb{R}^N,\ \lambda \in (0,1) \qquad \textbf{(LR)}.$$

Some recent results:

- Fujita-Ishii-Loreti (2006)
- Fujita-Loreti (2009)
- Bardi-Cesaroni-Ghilli (2015).

Application to ergodic peoblem

We consider the ergodic control problem

$$c - \mathrm{tr}(\sigma(x)\sigma^T(x)D^2v(x)) + \alpha x.Dv(x) + H(x, Dv(x)) = f(x) \qquad \text{in } \mathbb{R}^N, \qquad \textbf{(EP)}$$

where the unknown is a pair of a constant c and a function v.

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a) Answer question 1.

Theorem 2

Under the assumption of Theorem 1, there is a solution $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ of **(EP)**. *Let $(c_1, v_1), (c_2, v_2) \in \mathbb{R} \times C(\mathbb{R}^N)$ are two solutions of* **(EP)**, *then $c_1 = c_2$ and there is a constant $C \in \mathbb{R}$ such that $v_1 - v_2 = C$ in $\mathbb{R}^N$.*

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Proof:
1. Existence of solution. We first prove that there exists a constant $C > 0$ independent of λ such that

$$|\lambda u^\lambda(x)| \leq C \ \text{ on balls of } \mathbb{R}^N. \qquad (8)$$

We consider

$$\max_{\mathbb{R}^N}\{u^\lambda(x) - \phi(x)\} = u^\lambda(y) - \phi(y), \text{ for some } y \in \mathbb{R}^N.$$

Proof (Cont.)

Since u^λ is a viscosity solution and hence subsolution of **(SP)**. Then, at the maximum point y, we have

$$\lambda u^\lambda(y) - \mathrm{tr}(\sigma(y)\sigma^T(y)D^2\phi(y)) + \alpha y.D\phi(y) + H(y, D\phi(y)) \leq f(y).$$

Moreover, ϕ satisfies

$$-\mathrm{tr}(\sigma(y)\sigma^T(y)D^2\phi(y)) + \alpha y.D\phi(y) - C_H|D\phi(y)| \geq \phi(y) - B,$$

here $C_H, B > 0$. Therefore, using the sublinearity of H we obtain

$$\lambda u^\lambda(y) \leq f(y) - \phi(y) + B + C_H.$$

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Since y is a maximum point of $u(x) - \phi(x)$ for $x \in \mathbb{R}^N$, then using the above inequality we have

$$\begin{aligned} \lambda u^\lambda(x) &\leq \lambda\phi(x) + \lambda u^\lambda(y) - \lambda\phi(y) \quad \forall x \in \mathbb{R}^N \\ &\leq \lambda\phi(x) - \phi(y) + B + C_H + f(y) - \lambda\phi(y) \quad \lambda \in (0,1) \\ &\leq \phi(x) + B + C_H + f(y) - \phi(y) \\ &\leq C \qquad \forall x \in \mathbb{R}^N. \end{aligned}$$

The proof for the opposite inequality is the same by considering $\min_{\mathbb{R}^N}\{u^\lambda(x) + \phi(x)\}$.

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&\leq C \qquad \forall x \in \mathbb{R}^N.
\end{aligned}$$

The proof for the opposite inequality is the same by considering $\min_{\mathbb{R}^N}\{u^\lambda(x) + \phi(x)\}$. Now we set $v^\lambda(x) = u^\lambda(x) - u^\lambda(0)$ and using **(LR)** we have

$$|v^\lambda(x)| \leq C|x|(\phi(x) + 1)$$

$$|v^\lambda(x) - v^\lambda(y)| \leq C|x - y|(\phi(x) + \phi(y)).$$

Proof (Cont.)

Therefore $\{v^\lambda\}_{\lambda\in(0,1)}$ is a uniformly bounded and equi-continuous family on any balls of $\mathbb{R}^N$. Then by Ascoli's theorem, we can choose a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,1)$ such that

$$v^{\lambda_j} \to v \text{ in } C(\mathbb{R}^N) \text{ as } \lambda_j \to 0.$$

And from (8) we have

$$\lambda_j u^{\lambda_j}(x) \to c \in \mathbb{R} \quad \text{uniformly on balls of } \mathbb{R}^N.$$

Note that v^{λ_j} is a solution of

$$\begin{aligned} \lambda_j v^{\lambda_j} - \mathrm{tr}(\sigma(x)\sigma^T(x)D^2 v^{\lambda_j}(x)) + \alpha x.Dv^{\lambda_j}(x) + H(x, Dv^{\lambda_j}(x)) &= f(x) - \lambda_j u^{\lambda_j}(0) \\ \downarrow \lambda_j \to 0 \qquad & \\ -\mathrm{tr}(\sigma(x)\sigma^T(x)D^2 v) + \alpha x.Dv(x) + H(x, Dv(x)) &= f(x) - c. \end{aligned}$$

By stability of viscosity solutions we find that (c, v) is a solution of ergodic problem **(EP)**.
2. Uniqueness. See in Fujita-Ishii-Loreti (2006).

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b) Answer question 2.

Theorem 3

Let $u\in C(\mathbb{R}^N\times[0,T))$ be a solution of **(CP)** *with initial data $u(x,0)=u_0(x)$. Let $(c,v)\in\mathbb{R}\times C(\mathbb{R}^N)$ be a solution of* **(EP)**. *Let $L>0$ such that $|u_0(x)-v(x)|\le L$ in $\mathbb{R}^N$. Then*

$$\frac{u(x,t)}{t}\to c, \text{ as } t\to\infty.$$

Proof.

We easily check that $v(x) + ct - L$ and $v(x) + ct + L$ are solutions of **(CP)** with the initial condition replaced $v(x) \pm L$. Then applying comparison principle for **(CP)** we get

$$v(x) + ct - L \leq u(x,t) \leq v(x) + ct + L.$$

This implies

$$\frac{v(x) - L}{t} \leq \frac{u(x,t)}{t} - c \leq \frac{v(x) + L}{t}.$$

Sending t to ∞ we obtain the result.
Remark: More precise result-Long time behavior.
Let u is a solution of **(CP)**, (c, v) is a solution of **(EP)**. There exists a constant $a \in \mathbb{R}$ such that

$$u(x,t) - (ct + v(x)) \to a \ \text{ as } t \to \infty$$

locally uniformly in $\mathbb{R}^N$.

Proof of Theorem 1

Let $\delta, A, C_1 > 0$ and a C^2 concave and increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ which is constructed as following

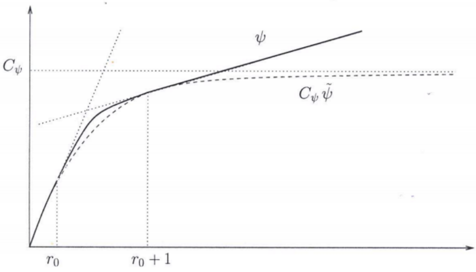

FIGURE 1. The function ψ

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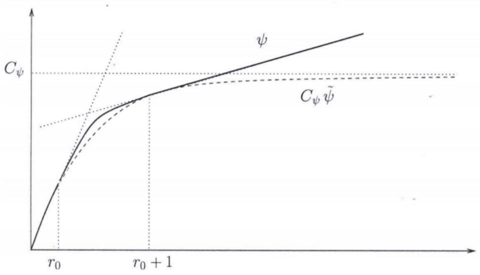

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We consider

$$M_{\delta,A,C_1} = \sup_{x,y\in\mathbb{R}^N} \left\{ u^\lambda(x) - u^\lambda(y) - \sqrt{\delta} - C_1(\psi(|x-y|) + \delta)(\phi(x) + \phi(y) + A) \right\}. \qquad (9)$$

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If $M_{\delta,A,C_1} \leq 0$ for some good choice of A, C_1, ψ independent of $\delta > 0$, then we get **(LR)** by letting $\delta \to 0$.

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$$\Phi(x,y) = C_1(\phi(x) + \phi(y) + A),$$
$$\varphi(x,y) = \sqrt{\delta} + (\psi(|x-y|) + \delta)\Phi(x,y).$$

Suppose that the supremum is achieved at some point (x, y) with $x \neq y$. Since u^λ is a viscosity solution of **(SP)**. In view of viscosity theory in Crandall-Ishii-Lions(1992), there exist $X, Y \in S^N$ such that the following viscosity inequality holds

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$$\begin{aligned} & \lambda(u^\lambda(x) - u^\lambda(y)) - (\mathrm{tr}(\sigma(x)\sigma^T(x)X) - \mathrm{tr}(\sigma(y)\sigma^T(y)Y)) + \alpha x.D_x\varphi - \alpha y.(-D_y\varphi) \\ & +H(x, D_x\varphi) - H(y, -D_y\varphi) \\ \leq \; & f(x) - f(y). \end{aligned} \tag{10}$$

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Since $M_{\delta,A,C_1} > 0$ and using the assumptions (4), (5), (6) and (7) to estimate for all the different terms, inequality (10) becomes

Proof of Theorem 1(Cont.)

$$\lambda\sqrt{\delta} - 4\nu\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y| \leq (2C_H + (1+\mu)\mathcal{C}_\sigma)\psi'(|x-y|) + 2(\psi(|x-y|)+\delta) + 2C_H + C_f|x-y|. \quad (11)$$

We droped Φ-terms on both sides of the above inequality to simplify of explaination.

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- *Case 1:* $r := |x-y| \leq r_0$. The function ψ above was chosen to be strictly concave for small $r \leq r_0$, we take profit of the ellipticity of the equation "$-4\nu\psi$"(r)" in (11) to control all the others terms.

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Goal: reach a contradiction in (11).

- *Case 1:* $r := |x-y| \leq r_0$. The function ψ above was chosen to be strictly concave for small $r \leq r_0$, we take profit of the ellipticity of the equation "$-4\nu\psi''(r)$" in (11) to control all the others terms.
- *Case 2:* $r := |x-y| \geq r_0$. In this case, the second derivative $\psi''(r)$ of the increasing concave function ψ is small and the ellipticity is not powerful enough to control the bad terms. Instead, we use the positive term $\alpha\psi'(r)r$ coming from the Ornstein-Uhlenbeck operator to control everything.

Note that by using the good terms as explained in the two above cases, we can choose all of parameters independently of λ to reach a contradiction in (11).

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Thank you for your attention!