Dynamical strategic influence in a social Network

Xavier Venel
(PSE, CES, University Paris 1 Panthéon-Sorbonne)

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Outline

DeGroot's model (1)

We introduce a model where an opinion is a real number in [0,1].

Definition

An opinion propagation model is described by

- a set of non-strategic agents denoted by K,
- a row-stochastic matrix M.

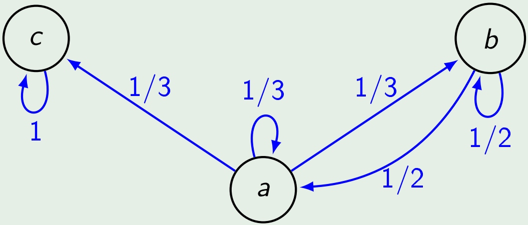
Example

We consider the set $K = \{a, b, c\}$ and

$$M = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

DeGroot's model (2)

Example



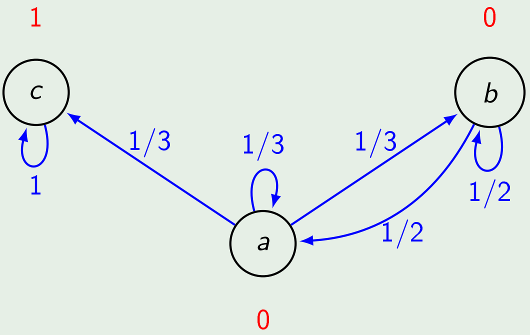
How does the opinion propagates:

- at stage 1, agent k has an opinion $p_1(k) \in [0,1]$,
- at stage $t+1$, agent k updates his opinion depending on the opinion of his neighbors at stage t:

$$p_{t+1}(k) = \sum_{l \in K} M_{kl} p_t(l).$$

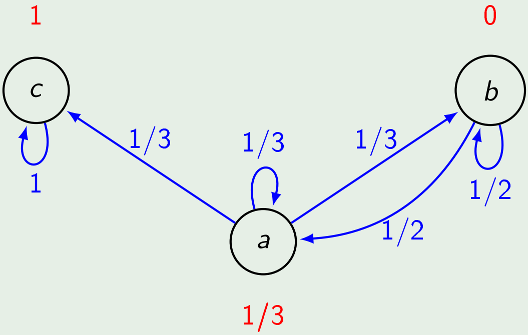
Example: t=1

Example



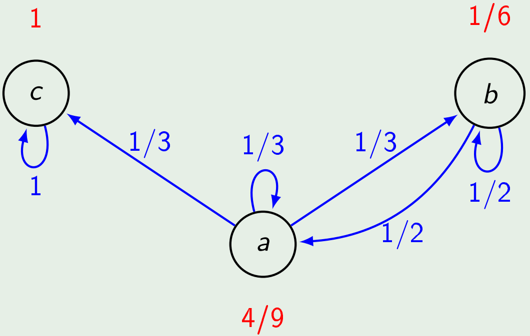
Example: t=2

Example



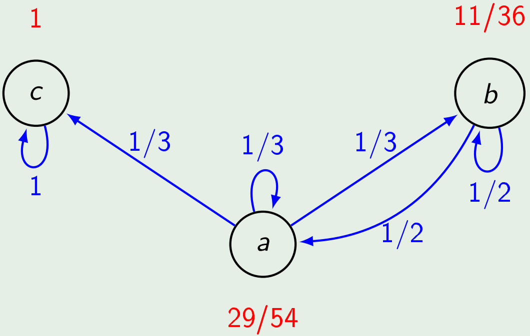
Example: t=3

Example



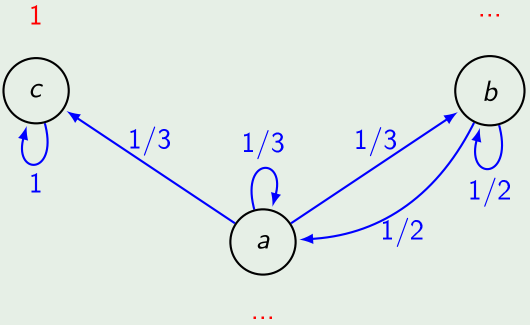
Example: t=4

Example



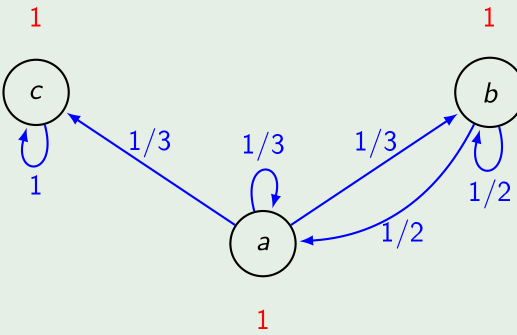
Example: t=??

Example



Example: $t = \infty$

Example



Definition

A matrix M is primitive if there exists $m \in I\!N$ such that every coefficient of M^m is strictly positive

Theorem (DeGroot 1974)

If M is primitive, then there is emergence of consensus in the society.

Example

For example,

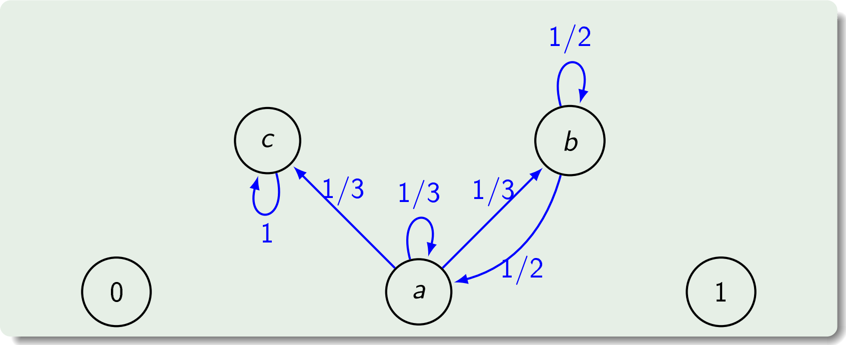
- if the network is connected,
- at least one player listens to himself.

A one-shot strategic game (Grabisch et al. 2015)

Definition

A static opinion game $G = (K, M, \lambda, \mu, p_1)$ is defined by

- a set K of non-strategic agents,
- a row-stochastic matrix M,
- λ a real number representing the lobbying power of agent 1,
- μ a real number representing the lobbying power of agent 0,
- an initial vector of opinion $p_1 \in [0,1]^K$


1/2
c
b
1/3
1/3
1/3
1
1/2
0
a
1

How the game is played:

- Player 1 and player 2 chooses respectively $i \in K$ and $j \in K$,
- It induces an opinion progation model of size $K+2$ with matrix $M(i,j)$ (see next slides for the formal definition) .
- Let $M_\infty(i,j) = \lim_{n \to +\infty} M(i,j)^n$.
- The payoff of player 1 is the mean-average opinion in the society in the long run:

$$g(i,j) = \frac{1}{K} \sum_{k \in K} \left(M_\infty(i,j) \begin{pmatrix} 1 \\ 0 \\ p_1 \end{pmatrix} \right)_k .$$

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Definition of the opinion propagation model $M(i,j)$

Assume agent $k \in K$ has D neighbors in M:

- if $k \neq i$ and $k \neq j$,

$$p_{t+1}(k) = (Mp_t)_k$$

- if $k = i$ and $k \neq j$,

$$p_{t+1}(k) = \frac{\delta}{D+\delta} + \frac{D}{D+\delta}(Mp_t)_k$$

- if $k \neq i$ and $k = j$,

$$p_{t+1}(k) = \frac{\mu}{D+\mu}(Mp_t)_k$$

- if $k = i$ and $k = j$,

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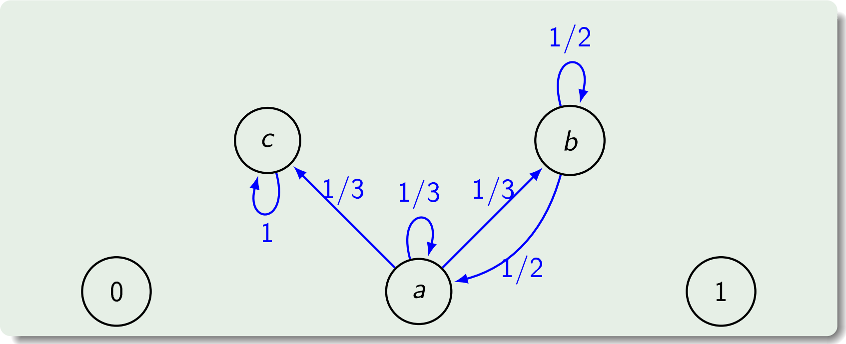
Example of M(i,j): $\delta = 2$, $\mu = 1$



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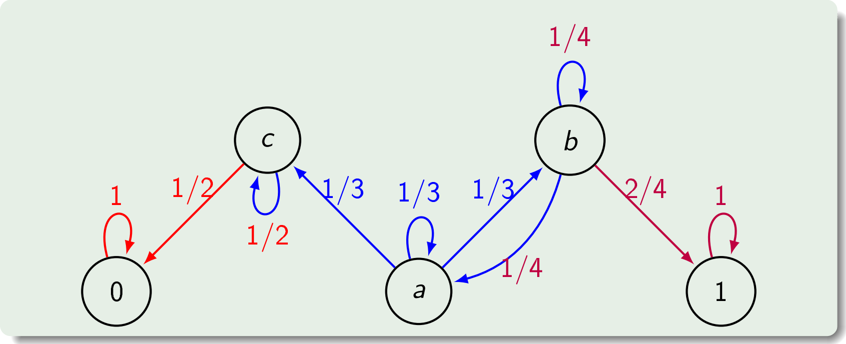


Results/Questions ?

Easy answer

There exists a value in mixed strategies.

Harder answers (Connected graph, Grabisch et al. 2015)

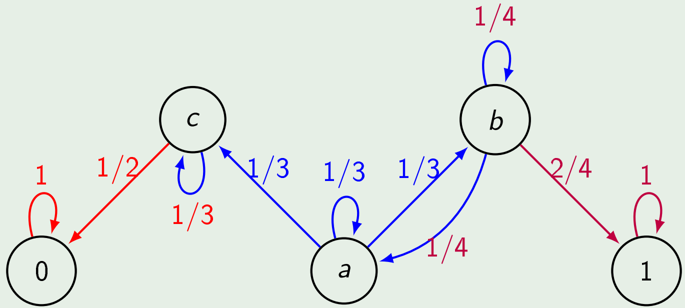
- Conditions for the existence of optimal pure strategies.
- Under the previous condition, caracterisation of the optimal strategies in terms of a centrality measure.

Outline

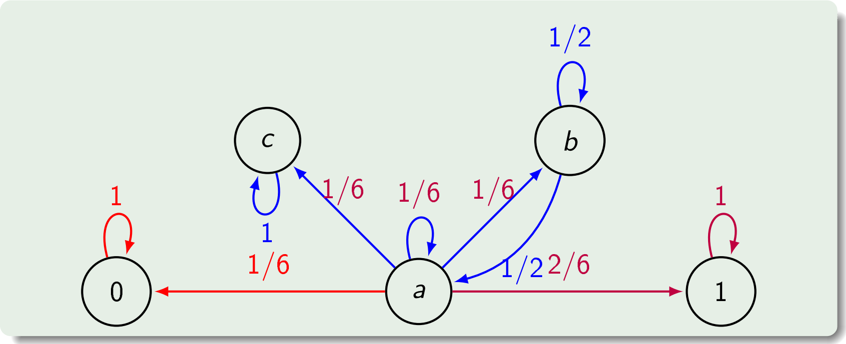
What if the strategic agents can change their target during the game?

- How can we model that?
- What solution concept can we use?
- Comparison with the static case (one/several connected component) ?

Example: t=1



Example: t=2



A stochastic model: $\Gamma = (X, I, J, q, g, p_1)$

Given (K, M, λ, μ), we define the stochastic game by:

- the set of states is $X = [0,1]^K$,
- the actions sets are $I = K$ and $J = K$,
- the transition $q : X \times I \times J \to X$ is defined such that

$$\begin{pmatrix} 1 \\ 0 \\ q(p,i,j) \end{pmatrix} = M(i,j) \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix},$$

- the stage payoff $g : Z \times I \times J \to [0,1]$ is defined by

$$g(p,i,j) = \frac{1}{K} \sum_{k \in K} p(k),$$

- an initial vector of opinion $p_1 \in [0,1]^K$.

How the game is played:

- At stage t, player 1 and player 0 chooses respectively $i_t \in K$ and $j_t \in K$,
- A new vector of opinion p_{t+1} is defined by

$$p_{t+1} = q(p_t, i_t, j_t),$$

- player 1 receives $g(p_t, i_t, j_t)$ whereas player 0 looses it.
- both players observe the actions played

Proposition

For every $i \in I$ and $j \in J$, q is 1-Lipschitz from $(X, \|.\|_\infty)$ to $(X, \|.\|_\infty)$.

Different criteria of evaluation:

- Average across time :

$$\gamma_n(p_1) = \mathbb{E}_{p_1,\sigma,\tau}\left(\frac{1}{n}\sum_{t=1}^{n} g(p_t)\right)$$

- Discounted across time:

$$\gamma_\lambda(p_1) = \mathbb{E}_{p_1,\sigma,\tau}\left(\lambda\sum_{t=1}^{+\infty}(1-\lambda)^{t-1} g(p_t)\right)$$

- Fixed date:

$$\gamma_\theta(p_1) = \mathbb{E}_{p_1,\sigma,\tau}(p_n)$$

Generalization of the static- approach

Definition

Let v be a real number,

- Player 1 guarantees v in $\Gamma(p_1)$ if

  $\forall \varepsilon > 0,\ \exists\ \sigma \in \Sigma,\ \exists N,\ \forall n \geq N,\ \forall \tau \in T,\ \gamma_n(p_1, \sigma, \tau) \geq v - \varepsilon.$
- Player 2 guarantees v in $\Gamma(p_1)$ if

  $\forall \varepsilon > 0,\ \exists \tau \in T,\ \exists N,\ \forall n \geq N,\ \forall \sigma \in \Sigma,\ \gamma_n(p_1, \sigma, \tau) \leq v + \varepsilon.$
- v is the uniform value of the game if both players can guarantee v.

Does there exists a uniform value in this type of games?

Not trivial.

Counter example with very
close assumptions by Ziliotto (2015)

Results

Theorem

Assume that the matrix M is primitive. Then the stochastic opinion game has a uniform value that does not depend on the original vector of opinions.

Theorem

Any stochastic opinion game has a uniform value.

Outline

Approximate by a game with a finite set of states.

Lemma

Let $S \subset K$ be a connected aperiodic component for M. There exists $\theta < 1$ and $m \in I\!N$ such that for all $p, p' \in X$,

$$\|q(p, i_1, j_1, \ldots, i_m, j_m) - q(p', i_1, j_1, \ldots, i_m, j_m)\|_{S,\infty} \leq \theta \|p - p'\|_{S,\infty},$$

if $i_1 \in S$ or $j_1 \in S$

- By approximating X by a finite grid there is an error of ε at every step.
- Due to the contraction, there is no accumulation of error.
    - First step: ε,
    - Second step: $\theta\varepsilon + \varepsilon$,
    - ...

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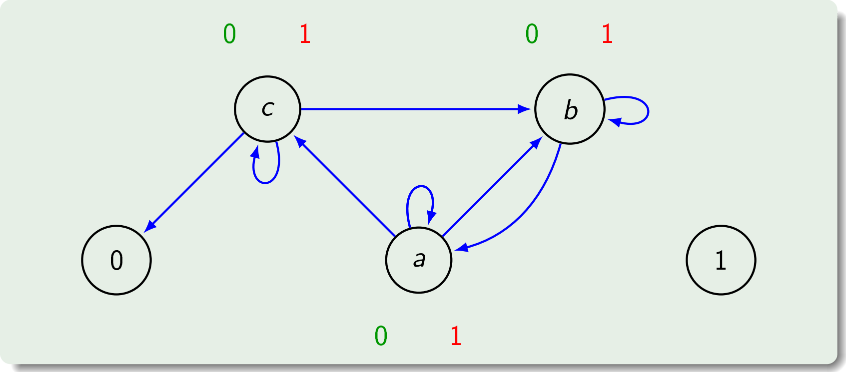
Intuition behind the Lemma

Opinion of the lobbies reach everyone in S.



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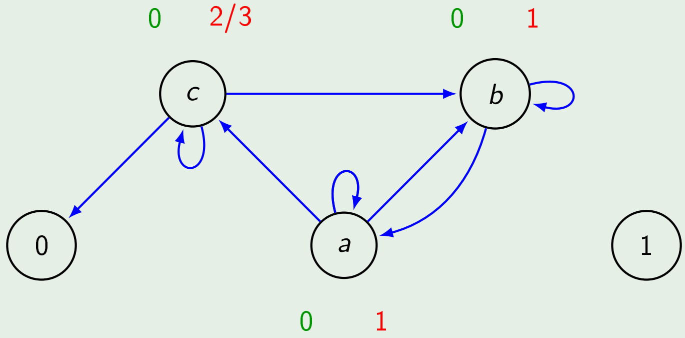

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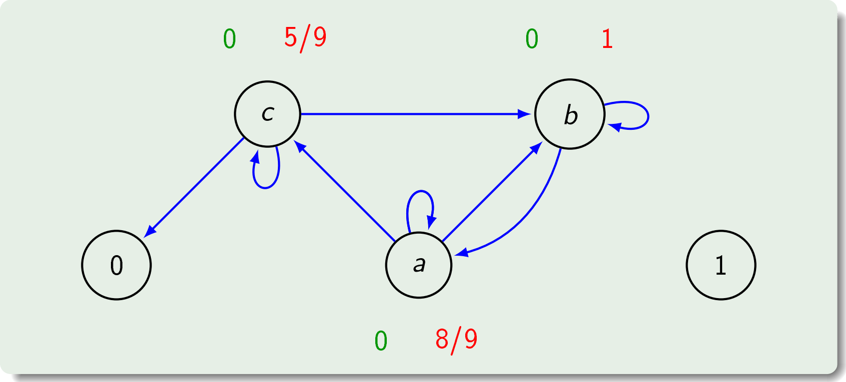

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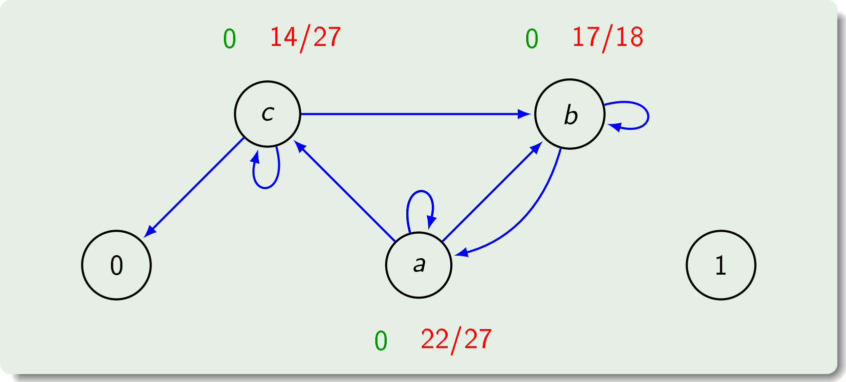
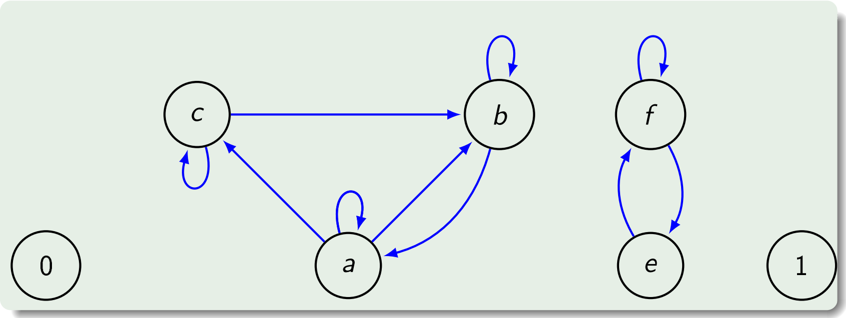


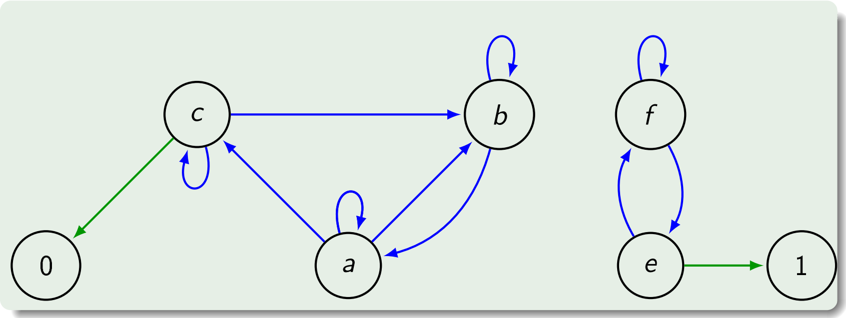
Need to regroup actions by blocks to obtain a contraction

With different ergodic class:



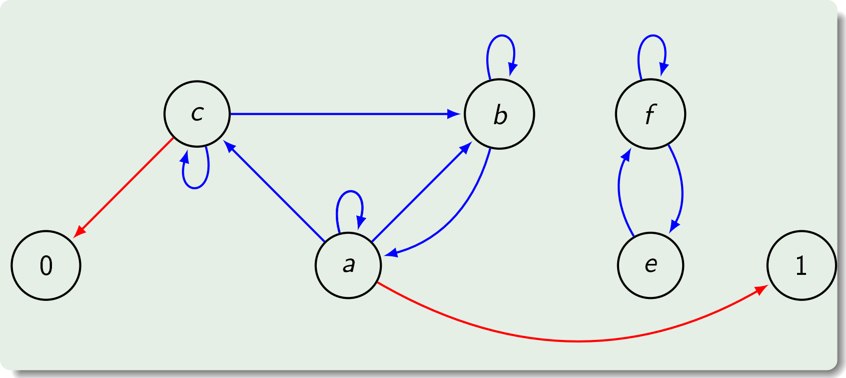
What are the differents cases possible?

With different ergodic class:



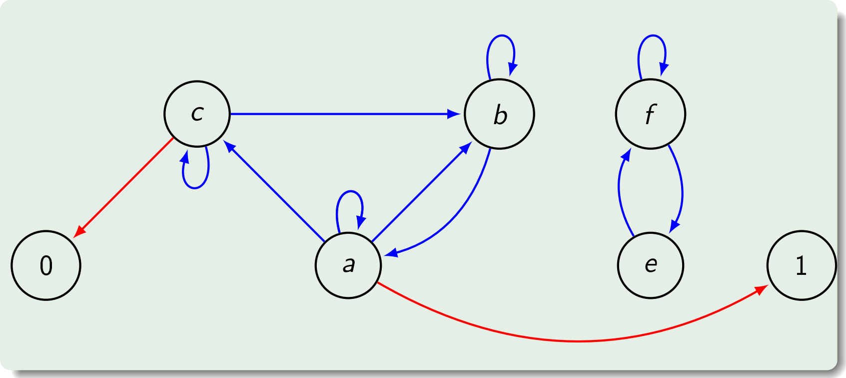
Contraction

With different ergodic class:



Only a contraction on the left part,

With different ergodic class:



Only a contraction on the left part, but the right part converges.

Ideas on how to prove the result?

- Let $(i,j) \in I \times J$, the state K of non-strategic agents is split into 2 sets:
  - states outside of the influence induced by (i,j), convergence,
  - states which are going to be influenced through some path. θ-contraction on the subspace

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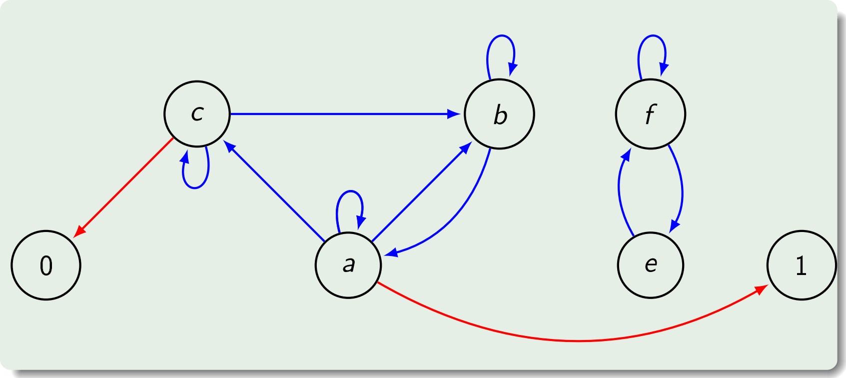
Block of fixed size is not sufficient

Let $m \in I\!N$ be a fixed length, $\tau = c^m$ and $\sigma = a^{m-1}e$:



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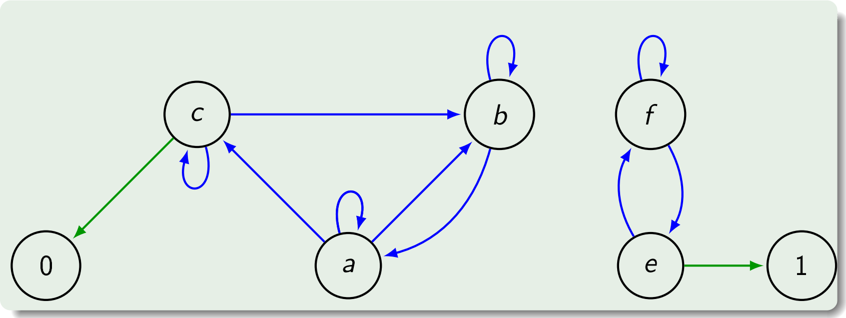

On the right-hand side, it is neither contracting nor converging.

Solution

Blocks of varying length: reset the timer each time a player target a new ergodic class.

Outline

Conclusions :

- Existence of the uniform value.
- Some stronger results for the aperiodic ergodic case.

Work in progress:

- What can we say if we consider a general model where $M(i,j)$ can be any row-stochastic model?
- Do strategic agent need to change target ? Maybe optimal strategies are simple.
- Characterization of the value (direct approach).

Further research:

- What happens if the strategic agents are not observing the actions played?
- Other modelisation of the opinion propagation.

Thanks