A characterization of the sets of equilibrium payoffs of finite games

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Sets of equilibrium payoffs of finite games

What is known about the set $E \subset \mathbb{R}^N$ of (mixed) equilibrium payoffs of some (one-shot) finite game with N players ?

- It is nonempty
- It is compact
- It is semialgebraic : finite union and intersection of sets each of the form $\{e \in \mathbb{R}^N, P(e_1, \cdots, e_N) < 0\}$ or $\{e \in \mathbb{R}^N, P(e_1, \cdots, e_N) \leq 0\}$.
- "Generically" it is finite and with odd cardinality.

Main problem

Answer to the question : $E \subset \mathbb{R}^N$ is the set of the equilibrium payoffs of some finite game with N players iff ...

- $N = 1$: iff E is a singleton.
- $N = 2$: iff E is the (nonempty) finite union of sets of the form $[a,b] \times [c,d]$ (Lehrer Solan Viossat '11)

What happens when $N \geq 3$?

Related literature

"Characterization" of sets of equilibria of finite games:

- Datta '03 : any real algebraic variety is isomorphic to the set of completely mixed equilibrium of some 3 player game, and to the set of completely mixed equilibrium of some M player binary game.
- Balkenborg-Vermeulen '14 : any nonempty connected compact semi-algebraic set is homeomorphic to one connected component of the set of equilibria of binary common interest game with payoffs 0 or 1.
- Levy '15, Viossat-V '15 : any nonempty compact semi-algebraic set is the projection of the set of equilibria of some game with M additional binary players.
- Levy '15 : any nonempty compact semi-algebraic set is the projection of the set of equilibria of some game with 3 additional nonbinary players.

Answer to our problem

Proposition

If $N \geq 3$, $E \subset \mathbb{R}^N$ is the set of equilibrium payoffs of some finite N-player game iff E is nonempty compact and semialgebraic.

Only the "if" part has to be proven.

Main result

We claim that this proposition is an easy consequence of the following result on equilibria:

Proposition

Let $N \geq 3$, *and* $E \subset [0,1[^N$ *be a nonempty compact semi algebraic set. Then there exists an* N*-player game* Γ, *and a particular action* X_*^i *for each player* i, *such that*
a) There exists an equilibrium which gives a probability e_i *to* X_*^i *for every* i *iff* $e = (e_1, \cdots, e_n) \in E$
b) All equilibria of Γ *have a payoff 0.*

Proof of the claim

Assume E is non empty compact semi algebraic and $E \subset [0,1[^N$, and consider the game Γ of the previous proposition.
Let Γ' be defined from Γ by adding 1 to the payoff of each player i iff player $i-1$ plays X_*^{i-1}.

- Γ' and Γ have the same set of equilibria
- Because of properties a) and b), the set of equilibrium payoffs of Γ' is $\{(e_N, e_1, \cdots, e_{N-1}) | (e_1, \cdots, e_N) \in E\}$
- One just has to relabel the players.

To deal with the case where $E \not\subset [0,1[^N$ one just has to apply affine transformations to the payoffs.

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Notations

- We assume $N = 3$ and
  $E = \{(e_1)^2 + (e_2)^2 + (e_3)^2 + e_1 e_2 + e_1 e_3 + e_2 e_3 \leq \frac{1}{2}\} \cap \mathbb{R}^3_+$.
- Actions will be denoted by uppercase (e.g. X^i_*) and probabilities by the corresponding lowercase (e.g. x^i_*).
- Payoff of an action of Player i will be given as "multiaffine" maps of the x^j for $j \neq i$ for example $g^1(X^1_*) = x^2_* + 2x^2_* x^3_*$.

Set of actions

Each player i has two family of actions:

- 11 Actions denoted with letter X : X^i_*, X^i_0, and $X^i_{j,k,l}$ for $1 \leq j+k+l \leq 2$. Called "unknowns". Give a payoff 0.
- Actions denoted with letter Y. Called "constraints".

We call an equilibrium nice if for each player all Y strategies are played with zero probability. Any nice equilibrium has payoff 0.

First part of the construction : nice equilibria

We will construct our game such that $e \in E$ iff there is a nice equilibrium with $x^i_* = e^i$ for all i.
Idea : in a nice equilibria, all strategies in Y are played with zero probability. If their payoff are defined as functions of the x^i, they will give some inequality constraints. We construct these constraints ensuring that $(x^1_*, x^2_*, x^3_*) \in E$.
Fix ε small enough (1/100 is ok).

The initialization constraints

Role : to ensure that at nice equilibria $x^i_{j,k,l} = \varepsilon(x^1_*)^j(x^2_*)^k(x^3_*)^l$ for $j+k+l=1$.
8 strategies for player 1 with payoffs

- $\pm(x^2_{0,1,0} - \varepsilon x^2_*)$
- $\pm(x^3_{0,1,0} - \varepsilon x^2_*)$
- $\pm(x^2_{0,0,1} - \varepsilon x^3_*)$
- $\pm(x^3_{0,0,1} - \varepsilon x^3_*)$

Similar strategies for player 2 and 3.

The induction constraints

Role : to ensure that at nice equilibria $x^i_{j,k,l} = \varepsilon(x^1_*)^j(x^2_*)^k(x^3_*)^l$ for $j+k+l=2$.

For example add strategies of player i with payoff

- $\pm(\varepsilon x^{i-1}_{2,0,0} - x^{i-1}_{1,0,0}x^{i+1}_{1,0,0})$, ensuring that $x^{i-1}_{2,0,0} = \varepsilon(x^1_*)^2$,
- $\pm(\varepsilon x^{i-1}_{1,1,0} - x^{i-1}_{1,0,0}x^{i+1}_{0,1,0})$, ensuring that $x^{i-1}_{1,1,0} = \varepsilon x^1_* x^2_*$,
- and so on.

The semialgebraic constraints

Role : to ensure that at nice equilibria $(x_*^1, x_*^2, x_*^3) \in E$.
For each player i add one strategy with payoff
$x_{2,0,0}^{i-1} + x_{0,2,0}^{i-1} + x_{0,0,2}^{i-1} + x_{1,1,0}^{i-1} + x_{1,0,1}^{i-1} + x_{0,1,1}^{i-1} - \frac{\varepsilon}{2}$
Because of the previous constraints, at equilibrium the payoff will be
$\varepsilon \left((x_*^1)^2 + (x_*^2)^2 + (x_*^3)^2 + x_*^1 x_*^2 + x_*^1 x_*^3 + x_*^2 x_*^3 - \frac{1}{2}) \right)$

Nice equilibria

The construction ensures that in any nice equilibria, $x_* \in E$.
Now if $x_* \in E$,

- Fix $x^i_{j,k,l} = \varepsilon (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l \geq 1$
- Fix $x^i_0 = 1 - x^i_* - \sum x^i_{j,k,l}$.
- If ε is small enough $x^i_0 \geq 0$ and the profile is well defined
- All constraints are nonprofitable by construction.

Hence there is a nice equilibrium with $x_* \in E$.

Second part of the construction : other equilibria

They may be a lot of other equilibria...
To get rid of them, add another constraint Y_*^i with payoff $K(1 - x_0^{i-1} - x_*^{i-1} - \sum x_{j,k,l}^{i-1})$ for K large.

- This payoff is 0 in any nice equilibrium, so this does not change the set of nice equilibria.
- If an equilibria is not nice, then $y_*^i = 1$ for all i.
- Hence in the unique not nice equilibrium, $x_* = (0,0,0) \in E$
- Payoff of this equilibrium is K and not 0, but that can be fixed by adding $-Ky_*^{i-1}$ to the payoff of each action of each Player i.

General case

In general we have to

- Deal with $N \geq 3$: easy fix
- Deal with arbitrary polynomials : easy fix
- Deal with intersections : easy fix
- Deal with unions : more difficult
- Deal with second part in general (when $0 \notin E$) : more difficult

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Reminder : Hilbert tenth problem

- Problem : "Find an algorithm to determine whether a given multivariate polynomial $P \in \mathbb{Z}[X_1, \cdots X_N]$ has a zero in $\mathbb{Z}^N$.
- MRDP Theorem, Matiyasevich '70 (using previous works by Robinson Davis and Putnam) : this is impossible, there is no such algorithm. "Undecidability of Hilbert tenth problem".
- Still undecidable even if one fix both N and the degree d of P (provided they are larger than explicit bounds).
- Problem is open if one replace "has a zero in $\mathbb{Z}^N$" by "has a zero in $\mathbb{Q}^N$" (Hilbert tenth problem on $\mathbb{Q}$).

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Consequences

Because of our construction, it is impossible to find an algorithm that, given a game with integer pure payoffs, answer to any of these questions:

- Is there an equilibrium in which the payoff of each player is the inverse of an integer ?
- Is there an equilibrium in which the probability that each player plays its first strategy is the inverse of an integer ?

Same impossibility even if one fix both the number of players and of actions (provided they are greater than some explicit bounds).

Other consequence : there exists an explicit game such that the answer to any of the two questions is yes iff Riemann's hypothesis is false.

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Consequences

If Hilbert tenth problem on $\mathbb{Q}$ is also undecidable, then it is impossible to find an algorithm that, given a game with integer payoffs answer to any of these questions:

- Is there an equilibrium in which the payoff of each player is a rational ?
- Is there an equilibrium in which the probability that each player plays its first strategy is a rational ?
- Is there an equilibrium in which all players play all strategies with some rational probability ?

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More questions

- Can we do such a construction with "robust" equilibria (without using weakly dominated strategies for example)?
- Can we use this to construct stochastic games with weird behavior ?
- What if we add some structure (games with perfect information for example) ?
- What about the set of payoffs of correlated equilibria ?

Thank you for your attention

Merci !