Stochastic homogenization of Hamilton-Jacobi equations: a counterexample.

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Stochastic Hamilton-Jacobi equation

Let $\epsilon > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and for $\omega \in \Omega$ consider the system

$$\begin{cases} \partial_t u_\epsilon(x,t,\omega) + H(Du_\epsilon(x,t,\omega), \frac{x}{\epsilon}, \omega) = 0 \qquad \text{in } \mathbb{R}^n \times (0,+\infty) \\ u_\epsilon(x,0,\omega) = 0 \qquad \text{in } \mathbb{R}^n \end{cases}$$

- H is continuous in (p,x) and Lipschitz in p, and $\lim_{\|p\| \to +\infty} H(p,x,\omega) = +\infty$ (coerciveness).
- The law of $\omega \to H(.,\omega)$ is assumed to be invariant by translation and ergodic.

$$\forall (x,t,\omega) \quad u_\epsilon(x,t,\omega) = \epsilon u_1\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}, \omega\right).$$

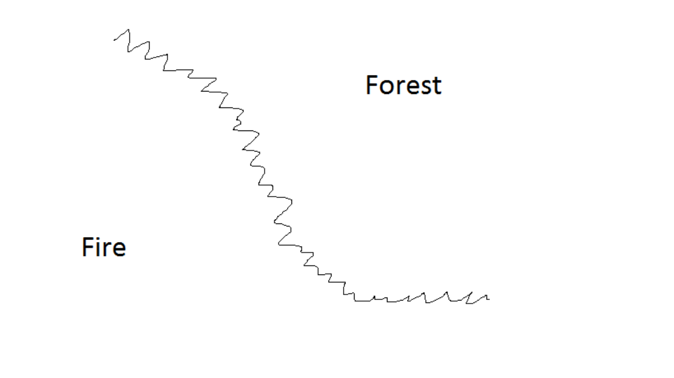
The system *homogenizes* if there exists $\bar{H}$ such that u_ϵ converges a.s. and uniformly in (x,t) to the solution of

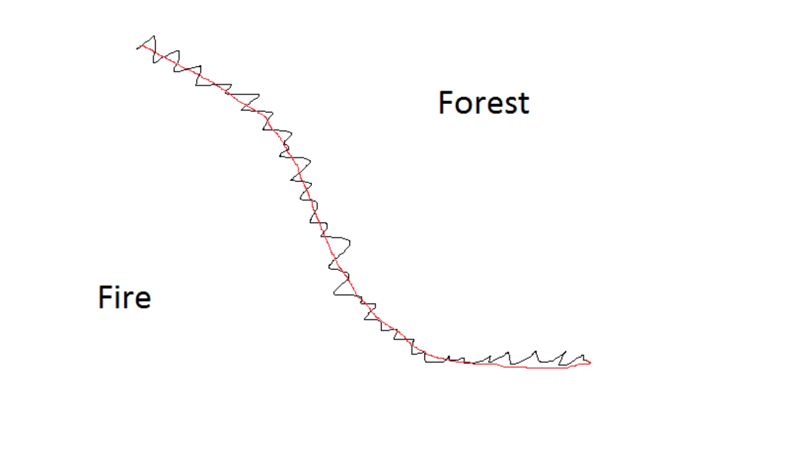
$$\begin{cases} \partial_t u(x,t) + \bar{H}(Du(x,t)) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = 0 & \text{in } \mathbb{R}^n \end{cases}$$

Example 1 : Fire front propagation

$$\forall (x,p) \in \mathbb{R}^2 \times \mathbb{R}^2 \quad H(x,p,\omega) := |p| \, h\left(x, \frac{p}{|p|}, \omega\right)$$

The fire front at time t corresponds to the level set $\left\{x \in \mathbb{R}^2 \mid u(x,t,\omega) = 0\right\}$.




Forest
Fire

Example 2 : Zero-sum stochastic differential games

- $A, B \subset \mathbb{R}^n$, $c : \mathbb{R}^n \times A \times B \times \Omega \to \mathbb{R}$,
  $f : \mathbb{R}^n \times A \times B \times \Omega \to \mathbb{R}^n$.
- Dynamics
  $$\dot{x}(t) = f(x(t), a(t), b(t), \omega),$$
  such that Player 1 controls the state (coerciveness)
- Payoff
  $$\gamma_T(a, b, \omega) := \frac{1}{T} \int_0^T c(x(t), a(t), b(t), \omega) dt.$$

Let $H(x, p, \omega) := \max_{a \in A} \min_{b \in B} \{-c(x, a, b, \omega) - p \cdot f(x, a, b, \omega)\}$

$u^\epsilon(0, 1, \omega)$ is the value of the game with duration $1/\epsilon$ and initial state 0.

Literature

Stochastic homogenization has been proven in the following cases :

- When H is periodic (consequence of Lions, Papanicolaou and Varadhan 1986)
- When H is convex in p (Souganidis 1999)
- When the law of H has finite range (Armstrong and Cardaliaguet 2015)

Question (Lions and Souganidis 2005, 2010, Kosygina 2007, Armstrong and Cardaliaguet 2015...) :

What happens in the general case ?

Theorem (Z.15)

There exists a stochastic Hamilton-Jacobi equation in $\mathbb{R}^2$ which does not homogenize.

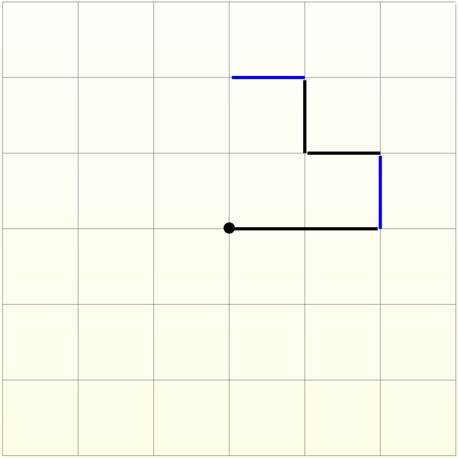
This equation comes from a zero-sum differential game, which itself is associated with a discrete-time zero-sum repeated game with state space $\mathbb{Z}^2$.

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- State space $\mathbb{Z}^2$,
- Cost function $c(., \omega)$ on the edges of $\mathbb{Z}^2$ (ergodic and $\mathbb{Z}^2$-invariant).

- The initial state is $(0, 0)$ and ω is publicly announced,
- Players play in turn,
- Player 1 moves the state along two edges, then Player 2 moves the state along one edge, etc.
- Player 1 (resp 2) minimizes (resp. maximizes)

$$\frac{1}{n} \sum_{m=1}^{n} c(e_m, \omega).$$

Jeu en 6 étapes



Denote by $v_n(\omega)$ the value of the *n*-stage game.

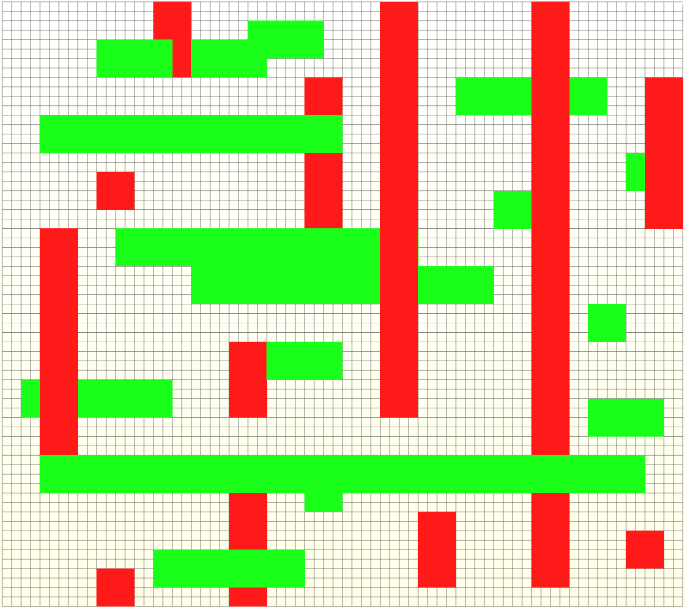
Theorem (Z. 15)

There exists $(\Omega, \mathcal{F}, \mathbb{P})$ and a cost function c such that $\mathbb{P}$-almost surely (v_n) does not converge.

This example can be adapted into a zero-sum differential game, which Hamilton-Jacobi equation does not homogenize.

Structure of c

- Put a cost 2 (huge cost) on the horizontal edges
- Let $T_k := 2^k$ ($k \geq 1$). Fill the space with two kinds of blocks :
  - rectangles of size $(10 \cdot T_k) \times 4$, $k \geq 1$, with cheap vertical edges (cost 1), called "green rectangles",
  - rectangles of size $4 \times (10 \cdot T_k)$, $k \geq 1$, with expensive vertical edges (cost 2), called "red rectangles".



Idea of the example

Consider $\Gamma_{T_k}(\omega)$ ($k \geq 1$) :

- If a green rectangle of size $(10T_k) \times 4$ is close to the origin : Player 1 forces the state to go to the green rectangle, and stay there at a cheap cost.

- If a red rectangle of size $4 \times (10T_k)$ is close to the origin : Player 2 plays horizontally towards the red rectangle, and forces Player 1 to play horizontally.

Construction of the green rectangles

For $k = 1$ to $+\infty$:

- For each $(l, m) \in \mathbb{Z}^2$, draw independently $X^k_{l,m} \sim B(2^{-k})$.

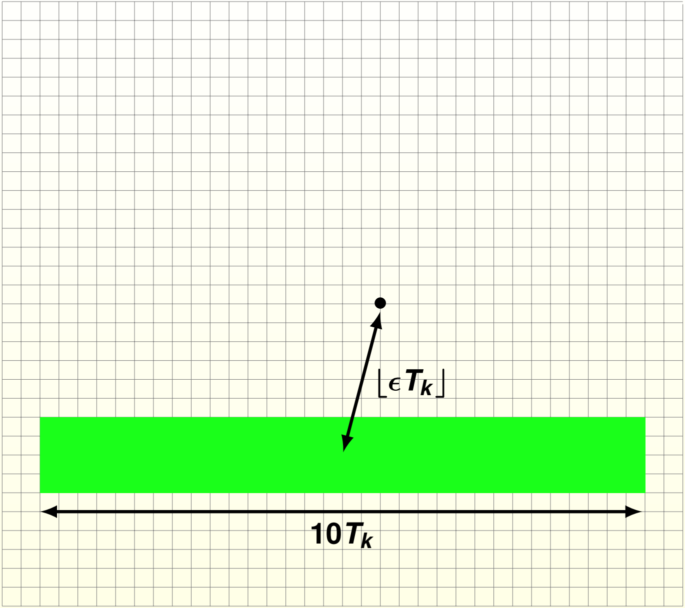
- For each $(l, m) \in \mathbb{Z}^2$ such that $X^k_{(l,m)} = 1$, create a green rectangle of size $(10T_k) \times 4$, centered on (l, m) :
  for each vertical edge e that lies in the rectangle, set $c(e, \omega) := 1$.

Construction of the red rectangles

For $k = 1$ to $+\infty$:

- For each $(l, m) \in \mathbb{Z}^2$, draw independently $Y^k_{(l,m)} \sim B(2^{-k})$.
- For each $(l, m) \in \mathbb{Z}^2$ such that $Y^k_{(l,m)} = 1$, create a red rectangle of size $4 \times 10T_k$, that is, for each vertical edge e that lies in the rectangle, proceed as follows :
  - If e lies in a green rectangle of size $(10T_{k'}) \times 4$, $k' \geq k$, do nothing.
  - Otherwise, set $c(e, \omega) := 2$.

The good scenario for Player 1



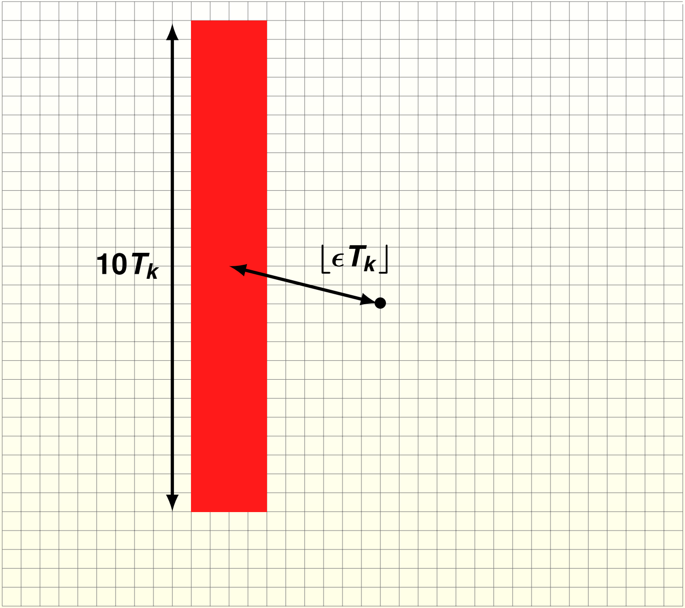
The good scenario for Player 1

Fix $\epsilon > 0$. For $k \geq 1$, let A_k be the event

"There exists a complete green rectangle of size $(10 \cdot T_k) \times 4$ at a distance smaller or equal to $\lfloor \epsilon T_k \rfloor$ from the origin".

- $\liminf_{k \to +\infty} \mathbb{P}(A_k) > 0$
- $\mathbb{P}$-a.s., there exists $(n_k(\omega))_{k \geq 1}$ such that A_{n_k} is realized.
- $v_{n_k}(\omega) \leq 4/3 + \epsilon$
-
$$\liminf_{n \to +\infty} v_n \leq \frac{4}{3} \quad \mathbb{P} - a.s.$$

The bad scenario for Player 1



The bad scenario for Player 1

For $k \geq 1$, let B_k be the event

"There exists a complete red rectangle of size $4 \times (10T_k)$ at a distance smaller or equal to $\lfloor \epsilon T_k \rfloor$ from the origin."

- $\liminf_{k \to +\infty} \mathbb{P}(B_k) > 0$
- $\mathbb{P}$-a.s., there exists $(n'_k(\omega))_{k \geq 1}$ such that $B_{n'_k}$ is realized.
- $v_{n'_k}(\omega) \geq 5/3 - \epsilon$
- $$\limsup_{n \to +\infty} v_n \geq 5/3 \quad \mathbb{P} - a.s.$$

A zero-sum differential game

- State space $\mathbb{R}^2$, control sets $A = B = [-1, 1]^2$.
- It is easy to smooth the discrete cost functions $c(., \omega)$ into 1-Lipschitz cost functions $\widetilde{c}(., \omega) : \mathbb{R}^2 \to [1, 2]$.
- Let $l : \mathbb{R}^2 \times [-1, 1]^2 \to [1, 2]$ defined by

$$\forall (x, a, \omega) \in \mathbb{R}^2 \times [-1, 1]^2 \times \Omega, \quad l(x, a, \omega) := \widetilde{c}(x, \omega) + 2|a_1|.$$

- Define $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \Omega$ by $\forall (p, x, \omega) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \Omega$

$$H(p, x, \omega) := \max_{a \in [-1,1]^2} \min_{b \in [-1,1]^2} \{-l(x, a, \omega) - p \cdot (2a + b)\}.$$

Theorem (Z.15)

$$\liminf_{\epsilon \to 0} u_\epsilon(0,1,\omega) \neq \limsup_{\epsilon \to 0} u_\epsilon(0,1,\omega) \quad \mathbb{P}\text{-a.s.}$$

Future research

Is it possible to use the discrete-time problem to prove the following positive result for the PDE problem :

If the law of the Hamiltonian does not correlate distant regions of space, then the HJ equation homogenizes.

- Finding an optimal strategy for Player 1 in the discrete-time problem can help building a supersolution of the Hamilton-Jacobi equation of the zero-sum differential game.
- Under mild assumptions, any Hamilton-Jacobi equation can be represented by a zero-sum differential game (Evans and Souganidis 1984).

Thank you for your attention !